Remark 1. It is a **very** rough partial draft. It is meant to express a rough research idea, not to be correct, readable, or complete. First read the book:

http://www.mathematics21.org/algebraic-general-topology.html upon which this formalistic is based (especially about the definition of generalized limit).

See also http://planetmath.org/MetasingularNumbers

The idea is simple (for these who know funcoids theory). But to find exact formulations about this is notoriously difficult. Below are attempts to formulate things about the theory of singularities.

New theory

Definition 2. *Singularity level* is a transitive, T_2-separable endofuncoid.

Let ν be a singularity level. Let Δ be a filter.

Define $\operatorname{SLA}(\nu)$ as:

$\operatorname{Ob}\operatorname{SLA}(\nu) = \{\nu \circ f \mid f \text{ is a monovalued funcoid with domain } \Delta\}$

$X\,[\operatorname{SLA}(\nu)]^*\,Y \Leftrightarrow \exists x \in X\,\forall K \in \operatorname{GR} x\,\exists L \in Y\colon L \sqsubseteq K$ [FIXME: It is probably not a funcoid.]

Remark 3. $\operatorname{GR} x$ is used despite of it is a funcoid not reloid.

Proposition 4. $\operatorname{SLA}(\nu)$ is an endofuncoid.

Proof. $\neg(\emptyset\,[\operatorname{SLA}(\nu)]^*\,Y)$ and $\neg(X\,[\operatorname{SLA}(\nu)]^*\,\emptyset)$ are obvious.

$I \cup J\,[\operatorname{SLA}(\nu)]^*\,Y \Leftrightarrow I\,[\operatorname{SLA}(\nu)]^*\,Y \vee J\,[\operatorname{SLA}(\nu)]^*\,Y$ is obvious.

$X\,[\operatorname{SLA}(\nu)]^*\,I \cup J \Leftrightarrow \exists x \in X\,\forall K \in \operatorname{GR} x\,\exists L \in I \cup J\colon L \sqsubseteq K \Leftrightarrow \exists x \in X\,\forall K \in \operatorname{GR} x\colon (\exists L \in I\colon L \sqsubseteq K \vee \exists L \in J\colon L \sqsubseteq K)$??

?? □

Alternative definition: [FIXME: It is probably not a funcoid.]

Definition 5. $X\,[\operatorname{SLA}(\nu)]^*\,Y \Leftrightarrow \exists z \in \operatorname{Ob}\mu\,\forall K \in \operatorname{GR} z\,\exists x \in X, y \in Y\colon x, y \sqsubseteq K$

Proposition 6. $\operatorname{SLA}(\nu)$ is a funcoid.

Proof. $X\,[\operatorname{SLA}(\nu)]^*\,Y \Leftrightarrow \exists z \in \operatorname{Ob}\mu, x \in X, y \in Y\,\forall K \in (\operatorname{GR} z)^{X \times Y}\colon x, y \sqsubseteq K_{x,y} \Leftrightarrow$??

$I \cup J\,[\operatorname{SLA}(\nu)]^*\,Y \Leftrightarrow \exists z \in \operatorname{Ob}\mu\,\forall K \in \operatorname{GR} z\,\exists x \in I \cup J, y \in Y\colon x, y \sqsubseteq K \Leftrightarrow \exists z \in \operatorname{Ob}\mu\,\forall K \in \operatorname{GR} z\colon (\exists y \in Y\colon y \sqsubseteq K \wedge (\exists x \in I\colon x \sqsubseteq K \vee \exists x \in J\colon x \sqsubseteq K)) \Leftrightarrow \exists z \in \operatorname{Ob}\mu\,\forall K \in \operatorname{GR} z\colon ((\exists y \in Y\colon y \sqsubseteq K \wedge \exists x \in I\colon x \sqsubseteq K) \vee (\exists y \in Y\colon y \sqsubseteq K \wedge \exists x \in J\colon x \sqsubseteq K)) \Leftrightarrow$?? □

Proposition 7. $\operatorname{SLA}(\nu)$ is T_2-separable.

Proof. ?? □

Proposition 8. $\operatorname{SLA}(\nu)$ is transitive.

Proof. ?? □

Galufuncoids

Let $\mathfrak{A}$ and $\mathfrak{B}$ are **Rel**-morphisms. I will denote like $(\sim_{\mathfrak{A}}) = \operatorname{GR}\mathfrak{A}$ and $(\sim_{\mathfrak{B}}) = \operatorname{GR}\mathfrak{B}$.

Definition 9. *Galufuncoids* between $\mathfrak{A}$ and $\mathfrak{B}$ is a quadruple $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)$ such that

$$\forall x \in \operatorname{Ob}\mathfrak{A}, y \in \operatorname{Ob}\mathfrak{B}: (\alpha x \sim_{\mathfrak{B}} y \Leftrightarrow x \sim_{\mathfrak{A}} \beta y).$$

Definition 10. $x\,[f]\,y \Leftrightarrow x \sim_{\operatorname{Src} f} \beta y$.

Obvious 11. $x\,[f]\,y \Leftrightarrow x \sim_{\operatorname{Src} f} \beta y \Leftrightarrow \alpha x \sim_{\operatorname{Dst} f} y$.

Remark 12. Galufuncoids are a generalization of both (pointfree) funcoids and Galois connections.

Definition 13. The *reverse* galufuncoid is defined by the formula:

$$(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)^{-1} = (\mathfrak{B}; \mathfrak{A}; \beta; \alpha).$$

Proposition 14. Composition of (composable) galufuncoids is a galufuncoid.

Proof. $(\alpha_2 \circ \alpha_1)x \sim y \Leftrightarrow \alpha_2\,\alpha_1\,x \sim y \Leftrightarrow \alpha_1\,x \sim \beta_2\,y \Leftrightarrow x \sim \beta_1\,\beta_2\,y \Leftrightarrow x \sim (\beta_1 \circ \beta_2)y$. □

Obvious 15. Galufuncoids form a category (similarly to the category of pointfree funcoids).

Definition 16. On the set of galufuncoids is defined a preorder by the formula: $f \sqsubseteq g \Leftrightarrow [f] \subseteq [g]$.

Galufuncoidal product

Functional galufuncoid

Definition 17. *Functional galufuncoid* ν/Δ of ν through filter Δ is the endo-galufuncoid defined by the formulas:

$\operatorname{Ob}(\nu/\Delta) = \mathsf{FCD}(\operatorname{Base}(\Delta); \operatorname{Ob}\nu)$

$\langle \nu/\Delta \rangle f = \nu \circ f$ and $\langle (\nu/\Delta)^{-1} \rangle f = \nu^{-1} \circ f$

$f \sim_{\operatorname{Ob}(\nu/\Delta)} g \Leftrightarrow g^{-1} \circ f \sqsupseteq \operatorname{id}_{\Delta}^{\mathsf{FCD}}$

[TODO: Restrict to the special case $f = \nu \circ F$ to make it T_2.]

[TODO: $X\,[\operatorname{SLA}(f)]\,Y$ is defined as existence of $x \in X$ such that for every entourage of x there is $y \in Y$ which is a subfilter of this entourage.]

Obvious 18. $\sim_{\operatorname{Ob}(\nu/\Delta)}$ is a symmetric relation.

Proposition 19. This is really a galufuncoid and $f\,[\nu/\Delta]\,g \Leftrightarrow g^{-1} \circ \nu \circ f \sqsupseteq \operatorname{id}_{\Delta}$.

Proof. We need to prove

$$\langle \nu/\Delta\rangle f \sim_{\mathrm{Ob}(\nu/\Delta)} g \Leftrightarrow g^{-1}\circ\nu\circ f \sqsupseteq \mathrm{id}_\Delta \Leftrightarrow f \sim_{\mathrm{Ob}(\nu/\Delta)} \langle(\nu/\Delta)^{-1}\rangle g.$$

Really,

$\langle \nu/\Delta\rangle f \sim_{\mathrm{Ob}(\nu/\Delta)} g \Leftrightarrow \nu\circ f \sim_{\mathrm{Ob}(\nu/\Delta)} g \Leftrightarrow g^{-1}\circ\nu\circ f \sqsupseteq \mathrm{id}_\Delta$

$f \sim_{\mathrm{Ob}(\nu/\Delta)} \langle(\nu/\Delta)^{-1}\rangle g \Leftrightarrow f \sim_{\mathrm{Ob}(\nu/\Delta)} \nu^{-1}\circ g \Leftrightarrow g^{-1}\circ\nu\circ f \sqsupseteq \mathrm{id}_\Delta$ □

Remark 20. A way to come to the above formula

$$\forall x \in \mathrm{atoms}\,\Delta\colon fx\,[\nu]\,gx \Leftrightarrow \forall x \in \mathrm{atoms}\,\Delta\colon x\,[g^{-1}\circ\nu\circ f]\,x \Leftrightarrow g^{-1}\circ\nu\circ f \sqsupseteq \mathrm{id}_\Delta.$$

Hierarchy of singularities

Consider two endo-galufuncoids μ and ν. Values on $\mathrm{Ob}\,\mu$ will behave like arguments of functions, of $\mathrm{Ob}\,\nu$ like values of functions.

I call $\mathrm{SLA}(\mathrm{Ob}\,\mu)$ *singularity level above* $\mathrm{Ob}\,\mu$ the set of sets of funcoids $\nu\circ f|_{\langle\mu\rangle^*\{x\}}$ (or alternatively of limits $\mathrm{xlim}\, f|_{\langle\mu\rangle^*\{x\}}$) where f is a monovalued principal funcoid in $\mathsf{FCD}(\mathrm{Ob}\,\mu;\mathrm{Ob}\,\nu)$. [TODO: Maybe exclude the zero funcoid?]

Consider a galufuncoid ω defined by the formulas:

$\langle\omega\rangle f = \nu\circ f$ and $\langle\omega^{-1}\rangle f = f\circ\nu$ and $f\,[\omega]\,g \Leftrightarrow \exists x\in\mathrm{Ob}\,\nu\colon g^{-1}\circ f \sqsupseteq \mathrm{id}^{\mathsf{FCD}}_{\langle\mu\rangle^*\{x\}}$.

We need to prove

$\langle\omega\rangle x \sim_\omega g^{-1} \Leftrightarrow \exists x\in\mathrm{Ob}\,\nu\colon g^{-1}\circ(\sim_\nu)\circ f \sqsupseteq \mathrm{id}_{\langle\mu\rangle^*\{x\}}$

and ??

The first is equivalent to $(\sim_\nu)\circ f \sim_\omega g^{-1} \Leftrightarrow \exists x\in\mathrm{Ob}\,\nu\colon g^{-1}\circ(\sim_\nu)\circ f \sqsupseteq \mathrm{id}^{\mathsf{FCD}}_{\langle\mu\rangle^*\{x\}}$.

Really, $(\sim_\nu)\circ f \sim_\omega g^{-1} \Leftrightarrow \exists x\in\mathrm{Ob}\,\nu\colon g^{-1}\circ(\sim_\nu)\circ f \sqsupseteq \mathrm{id}_{\langle\mu\rangle^*\{x\}}$

Lemma 21. Let $\nu\circ\nu\sqsubseteq\nu$ and $\nu^{-1}\circ\nu\sqsubseteq\nu$. If x and y are ultrafilters, then $x\,[\nu]\,y \Leftrightarrow \langle\nu\rangle x = \langle\nu\rangle y$.

Proof. [TODO: Prove for the more general case of galufuncoids. It's problematic.]

$x\,[\nu]\,y \Leftrightarrow y \sqsubseteq \langle\nu\rangle x \Rightarrow \langle\nu\rangle y \sqsubseteq \langle\nu\circ\nu\rangle x \Rightarrow \langle\nu\rangle y \sqsubseteq \langle\nu\rangle x$. So taking symmetry into account we have $x\,[\nu]\,y \Rightarrow \langle\nu\rangle x = \langle\nu\rangle y$.

Let now $\langle\nu\rangle x = \langle\nu\rangle y$, Then $y \not\sqsubseteq \langle\nu^{-1}\circ\nu\rangle x$; $y \sqsubseteq \langle\nu^{-1}\circ\nu\rangle x$; $y \sqsubseteq \langle\nu\rangle x$; $y \not\sqsubseteq \langle\nu\rangle x$; $x\,[\nu]\,y$. □

Theorem 22. $f\,[\nu/\langle\mu\rangle^*\{x\}]\,g \Leftrightarrow \nu\circ f|_{\langle\mu\rangle^*\{x\}} = \nu\circ g|_{\langle\mu\rangle^*\{x\}}$ for $f,\,g\in\mathrm{SLA}(\mathrm{Ob}\,\mu)$. [TODO: Generalize it for any Δ instead of $\langle\mu\rangle^*\{x\}$?]

Proof. It's enough to prove $\langle f\rangle x\,[\nu]\,\langle g\rangle x \Leftrightarrow \langle\nu\rangle\langle f\rangle x = \langle\nu\rangle\langle g\rangle x$ for every ultrafilter x.

$\langle f\rangle x\,[\nu]\,\langle g\rangle x \Leftrightarrow \langle g\rangle x \sim_{\mathrm{Ob}\,\mu} \langle\nu\rangle\langle f\rangle x \Rightarrow ?? \Rightarrow \langle\nu\rangle\langle g\rangle x \sim_{\mathrm{Ob}\,\mu} \langle\nu\rangle\langle f\rangle$

?? □

Theorem 23. $\mathrm{SLA}(\mathrm{Ob}\,\mu)$ is T_2-separable.

The reloid ν' on the set $\mathrm{SLA}(\mathrm{Ob}\,\nu)$ could be defined by one of the two formulas below:

For every point x get

$$\sigma_f(x) = \left\{u \in \bigcup f \mid \operatorname{dom} u = \langle\mu\rangle^*\{x\}\right\}.$$

$q_f = \bigsqcup_{x \in \operatorname{Ob}\mu} \sigma_f(x)$.

$$f\,[\nu']\,g \Leftrightarrow \exists x \in \operatorname{Ob}\mu\colon \bigsqcup q_f\,[\nu/\langle\mu\rangle^*\{x\}] \bigsqcup q_g.$$

or

$$f\,[\nu']\,g \Leftrightarrow \nu \circ q_f = \nu \circ q_g.$$

[TODO: How to define galufuncoids corresponding to the above formulas (if at all possible)?]
$\exists x \in D\colon f\,[\nu(x)]\,g \Leftrightarrow \exists x \in D\colon g \not\asymp \langle\nu(x)\rangle f \Leftrightarrow ?? \Leftrightarrow g \not\asymp \bigsqcup_{x \in D} \nu(x)$.
The ?? does not generally hold: our lattices are co-brouwerian not brouwerian!

The above formula holds if g is a discrete reloids. So replace every funcoid $f \in \operatorname{SLA}(\operatorname{Ob}\nu)$ with $(\mathsf{RLD})_{\text{in}}\, f$. Then continue for arbitrary reloids.

Another try: $\langle\nu'\rangle x = \nu \circ \bigsqcup \bigcup x$

$y \not\asymp \langle\nu'\rangle x \Leftrightarrow y \not\asymp \nu \circ \bigsqcup \bigcup x \Leftrightarrow \nu \not\asymp y \circ (\bigsqcup \bigcup x)^{-1} \Leftrightarrow \nu^{-1} \not\asymp (\bigsqcup \bigcup x) \circ y^{-1}$ [FIXME: x and y are of different types.]

$x \not\asymp \langle\nu^{-1}\rangle y \Leftrightarrow x \not\asymp \nu^{-1} \circ \bigsqcup \bigcup y$

The rest

One more other definition:

$$f\,[\nu'']\,g \Leftrightarrow \left(\bigsqcup \bigcup g\right)^{-1} \circ \nu \circ \left(\bigsqcup \bigcup f\right) \neq 0$$

Yahoo! $(i \cup j)\,[\nu'']\,g \Leftrightarrow i\,[\nu'']\,g \vee j\,[\nu'']\,g$ etc.

Proof. $(i \cup j)\,[\nu'']\,g \Leftrightarrow (\bigsqcup \bigcup g)^{-1} \circ \nu \circ (\bigsqcup \bigcup (i \cup j)) \neq 0 \Leftrightarrow (\bigsqcup \bigcup g)^{-1} \circ \nu \circ ((\bigsqcup \bigcup i) \sqcup (\bigsqcup \bigcup j)) \neq 0 \Leftrightarrow (\bigsqcup \bigcup g)^{-1} \circ \nu \circ (\bigsqcup \bigcup i) \sqcup (\bigsqcup \bigcup g)^{-1} \circ \nu \circ (\bigsqcup \bigcup j) \neq 0 \Leftrightarrow (\bigsqcup \bigcup g)^{-1} \circ \nu \circ (\bigsqcup \bigcup i) \neq 0 \vee (\bigsqcup \bigcup g)^{-1} \circ \nu \circ (\bigsqcup \bigcup j) \neq 0 \Leftrightarrow i\,[\nu'']\,g \vee j\,[\nu'']\,g$ □

Proposition 24. ν'' is a galufuncoid.

Proof. ?? □

An attempt of an alternate definition:

$f\,[\nu^*]\,g \Leftrightarrow \operatorname{xlim} f \circ \nu \circ \operatorname{xlim} g \neq 0$ [FIXME: Does this make sense?] [TODO: Differentiate generalized limit as a set of funcoids or its variation as a funcoid-value function.]

Proposition 25. $\operatorname{xlim} f \in \operatorname{SLA}(\operatorname{Ob}\nu)$ if $f \in \mathsf{FCD}(\operatorname{Ob}\mu; \operatorname{Ob}\nu) \setminus \left\{0^{\mathsf{FCD}(\operatorname{Ob}\mu;\operatorname{Ob}\nu)}\right\}$.

Proof. ?? □

Proposition 26. $\tau(x) \in \operatorname{SLA}(\operatorname{Ob}\nu)$.

Proof. ?? □

Proposition 27. $\tau(x)\,[\nu']\,\tau(y) \Leftrightarrow x\,[\nu]\,y$.

Proof. ?? $\square$

Metasingular numbers

Let $y \in \mathrm{SLA}(\mathrm{Ob}\,\mu)$. I will denote $r(y)$ such $x \in \mathrm{Ob}\,\nu$ that $\tau(x) = y$, if such x exists.

I will call *base singular numbers* the set $\mathrm{BSN} = \mathrm{Ob}\,\nu \cup \mathrm{SLA}(\mathrm{Ob}\,\nu) \cup \mathrm{SLA}(\mathrm{SLA}(\mathrm{Ob}\,\nu)) \cup \ldots$.

[TODO: Set that this union is disjoint.]

I will call *meta-singular numbers* the set $\mathrm{MSN} = \{y \in \mathrm{SLA}(\mathrm{Ob}\,\nu) \mid \nexists x \in \mathrm{BSN}\colon y = \tau(x)\}$.

Definition 28. I call *reduced* BSN its corresponding MSN (that is r applied to our BSN a natural number of times while possible).

Definition 29. I call *reduced limit* the reduced generalized limit.

Functions with meta-singular numbers as arguments

Let f is an n-ary (n is an arbitrary possibly infinite index set) function on $\mathrm{Ob}\,\nu$. Then define function f' on $\mathrm{SLA}(\mathrm{Ob}\,\nu)$ as:

$$f'(b) = \left\{ g \circ \prod^{(A)} b \mid g \in f' \right\}.$$

We can't use cross-composition product instead of above sub-atomic product because cross-composition product is not a funcoid (just a pointfree funcoid). We can replace sub-atomic product with displaced product, but as about my opinion displaced product seems more weird an inconvenient.

The above induces a trivial definition of functions on MSN but only for functions of finite arity (because having a finite set of MSN we can raise them to the same (maximum) level).

On differential equations

Replacing limit in the definition of derivative with the above defined reduced limit, the base set $\mathrm{Ob}\,\mu$ with MSN and operations f on the set $\mathrm{Ob}\,\mu$ with corresponding operations on MSN, we get a new interpretation of a differential equation (DE) (ordinary or partial).

Let call such (enhanced) differential equations *meta-singular equations* (as opposed to *non-singular equations* that is customary differential equations).

There arise the following questions:

Definition 30. I call a solution of a DE a *trivial restriction* if it is a restriction (to the set of non-singular points) of exactly one enhanced DE.

We need to find when there are solutions which are not trivial restrictions.

Then we can split such non-trivial solutions into following classes:

- "added solutions" are solutions whose restriction to non-singularity points is not a non-singular solution;

- "alternate solutions" is when an non-singular solution is a restriction of more than one meta-singular solution;

- "disappearing solutions" when a non-singular solution is not a restriction of a meta-singular solution.

Special case of general relativity

I am not a expert in general relativity (I am not even a professional mathematician).

But it looks like that the equations of general relativity can be converted (as described above) into meta-singular equations. For the special case of general relativity equations, the above classes are:

- "added solutions" would possibly characterize a "world above" described not with real numbers as our world but with singularities. This may or may not be of physical interest.

- "alternate solutions" would characterize black (or white) holes with additional information hidden inside. This additional information may probably solve the well known paradox of information disappearing when it falls into a black hole.

- "disappearing solutions" would mean that the laws of nature are possibly more restrictive than considered in more traditional physics. Could it resolve time-machine related paradoxes?

I again repeat that I am not an expert in general relativity. I seek collaboration with general relativity experts to solve the problems I've formulated.

I think (except of the case of the negative result that is there are no non-trivial solutions) this research is destined to receive Nobel Prize and/or Fundamental Physics Prize. I want my half.

Note that the group G (see the definition of generalized limit in my book) for general relativity can be defined in two different ways: as the group of homeomorpisms of the curved space or as the group of only uniformly continuous (in both directions) bijections. This gives us **two** new theories of general relativity.