$$\forall X \in \prod_{i\in D} \operatorname{up} a_i, Y \in \prod_{i\in D} \operatorname{up} b_i \exists x \in \prod_{i\in D} \operatorname{atoms}{\uparrow} X_i, y \in \prod_{i\in D} \operatorname{atoms}{\uparrow} Y_i \exists N \in \mathcal{F} \forall j \in N : x_j \left[f_j\right] y_j$$

where $D = \operatorname{dom} f$.

Thus by the lemma $\exists N \in \mathcal{F} \forall i \in N : a_i \left[f_i\right] b_i$, that is $a \left[\prod^{[\mathcal{F}]} f\right] b$. □

FiXme: TODO: when $\Pr_j \prod^{[\mathcal{F}]}_{i\in D} a_i = a_j$?

1. More on product of reloids

FiXme: Move this to a more appropriate place.

Definition 2087. $\prod^{(Y)}_{i\in\operatorname{dom} f} f = \prod^{(A)}_{i\in\operatorname{dom} f} (\mathsf{FCD}) f$ for an indexed family f of reloids.

Proposition 2088.

$$a \left[\prod^{(Y)}_{i\in\operatorname{dom} f} f\right] b \Leftrightarrow \forall i \in \operatorname{dom} f : f_i \not\asymp \Pr^{\mathsf{RLD}}_i a \times^{\mathsf{RLD}} \Pr^{\mathsf{RLD}}_i b.$$

Proof. $f_i \not\asymp \Pr^{\mathsf{RLD}}_i a \times^{\mathsf{FCD}} \Pr^{\mathsf{RLD}}_i b \Leftrightarrow (\mathsf{FCD}) f_i \sqsupseteq \Pr^{\mathsf{RLD}}_i a \times^{\mathsf{FCD}} \Pr^{\mathsf{RLD}}_i b \Leftrightarrow a \left[(\mathsf{FCD}) f_i\right] b$. □

Example 2089. The funcoid p described by the formula (for atomic reloids a and b)

$$a\ p\ b \Leftrightarrow \forall i \in \operatorname{dom} f : f_i \sqsupseteq \Pr^{\mathsf{RLD}}_i a \times^{\mathsf{RLD}} \Pr^{\mathsf{RLD}}_i b$$

does not exist (in general), even if we restrict to 2-indexed families only.

Proof. For the case if $f = \llbracket v, w \rrbracket$ is a 2-indexed family of reloids, the formula which we need to disprove takes the form:

$$a\ p\ b \Leftrightarrow v \sqsupseteq \operatorname{dom} a \times^{\mathsf{RLD}} \operatorname{dom} b \wedge w \sqsupseteq \operatorname{im} a \times^{\mathsf{RLD}} \operatorname{im} b.$$

Take $v = w = 1^{\mathbf{Rel}}$ on an infinite set. Suppose for the contrary p exists and is a funcoid. Then

$$\forall X \in \operatorname{up} a, Y \in \operatorname{up} b \exists x \in \operatorname{atoms}{\uparrow} X, y \in \operatorname{atoms}{\uparrow} Y : x\ p\ y \Rightarrow a\ p\ b.$$

For a counter-example take $a = b$ to be a nontrivial ultrafilter. Then for every $X \in \operatorname{up} a$, $Y \in \operatorname{up} b$ take $x = y$ to be singletons on $X \cap Y$. We have $x\ p\ y$, but not $a\ p\ b$. □