CHAPTER 10

Product of funcoids over a filter

The following definition is inspired by the usual definition of Tychonoff product of topological spaces.

DEFINITION 2081. Let f be an indexed family of funcoids. Let $\mathcal{F}$ be a filter on $\operatorname{dom} f$.

$$a \left[\prod^{[\mathcal{F}]} f\right] b \Leftrightarrow \exists N \in \mathcal{F} \forall i \in N : \operatorname*{Pr}_i^{\mathsf{RLD}} a \; [f_i] \; \operatorname*{Pr}_i^{\mathsf{RLD}} b$$

for atomic reloids a and b.

REMARK 2082. We are especially interested in the special case when $\mathcal{F}$ is the cofinite filter. In this case $a \left[\prod^{[\mathcal{F}]} f\right] b$ is defined by the condition that $\operatorname{Pr}_i^{\mathsf{RLD}} a \; [f_i]$ $\operatorname{Pr}_i^{\mathsf{RLD}}$ for an infinite number of indexes i.

OBVIOUS 2083. $a \left[\prod^{\left[\top^{\mathscr{F}(\operatorname{dom} f)}\right]} f\right] b \Leftrightarrow a \left[\prod^{(A)} f\right] b$.

PROPOSITION 2084. $\neg(\mathcal{X} \; [f] \; \mathcal{Y})$ implies $\neg(X \; [f] \; Y)$ for some $X \in \operatorname{up} \mathcal{X}$, $Y \in \operatorname{up} \mathcal{Y}$.

PROOF. Suppose $\neg(\mathcal{X} \; [f] \; \mathcal{Y})$. Then $\mathcal{Y} \asymp \langle f \rangle \mathcal{X}$. Thus by separability of core for filters $Y \asymp \langle f \rangle \mathcal{X}$ for some $Y \in \operatorname{up} \mathcal{Y}$, that is $\neg(\mathcal{X} \; [f] \; Y)$. Apply this result twice. □

LEMMA 2085.

$$\forall X \in \prod_{i \in D} \operatorname{up} a_i, Y \in \prod_{i \in D} \operatorname{up} b_i \exists x \in \prod_{i \in D} \operatorname{atoms} \uparrow X_i, y \in \prod_{i \in D} \operatorname{atoms} \uparrow Y_i \exists N \in \mathcal{F} \forall j \in N : x_j \; [f_j] \; y_j$$

implies $\exists N \in \mathcal{F} \forall i \in N : a_i \; [f_i] \; b_i$.

PROOF. Suppose for the contrary $\neg(a_i \; [f_i] \; b_i)$ for all $i \in N$ where $N \in \mathcal{F}$ (i.e. for an infinite number of indexes if $\mathcal{F}$ is the cofinite filter). Then (lemma above) there are $X_i \in \operatorname{up} a_i$ and $Y_i \in \operatorname{up} b_i$ such that $\neg(X_i \; [f_j]^* \; Y_i)$ for $i \in N$. Thus $\neg(x_i \; [f_i] \; y_i)$ for $i \in N$, contrary to the condition. □

PROPOSITION 2086. The funcoid $\prod^{[\mathcal{F}]} f$ exists.

PROOF. We need to prove that

$$\forall X \in \operatorname{up} a, Y \in \operatorname{up} b \exists x \in \operatorname{atoms} \uparrow^{\mathsf{RLD}} X, y \in \operatorname{atoms} \uparrow^{\mathsf{RLD}} Y : x \left[\prod^{(A2)} f\right] y$$

implies $a \left[\prod^{[\mathcal{F}]} f\right] b$.

Equivalently transforming it: FiXme: More detailed proof.

$\forall X \in \operatorname{up} a, Y \in \operatorname{up} b \exists x \in \operatorname{atoms} \uparrow^{\mathsf{RLD}} X, y \in \operatorname{atoms} \uparrow^{\mathsf{RLD}} Y$
$\exists N \in \mathcal{F} \forall i \in N : \operatorname{Pr}_i^{\mathsf{RLD}} x \; [f_i] \; \operatorname{Pr}_i^{\mathsf{RLD}} y$;
$\forall X \in \operatorname{up} a, Y \in \operatorname{up} b \exists x \in \prod_{i \in \operatorname{dom} f} \operatorname{atoms} \uparrow^{\mathsf{RLD}} X_i, y \in \prod_{i \in \operatorname{dom} f} \operatorname{atoms} \uparrow^{\mathsf{RLD}} Y_i$
$\exists N \in \mathcal{F} \forall i \in N : x_i \; [f_i] \; y_i$;