The first easily follows from $\tilde{\phi} \circ i = \phi$ and the fact that $\tilde{\phi}$ preserves binary joins.

The second easily follows from $\tilde{\phi} \circ i = \phi$ and that ϕ preserves $\bot$.

The third follows from the fact that $\tilde{\phi}$ preserves joins. □

COROLLARY 2046. The poset of prestaroids $\mathsf{preStrd}(\mathfrak{A})$ is isomorphic to an ideal (on a join-semilattice), provided that $\mathfrak{A}$ is an indexed family of join-semilattices.

PROOF. $\mathsf{preStrd}(\mathfrak{A}) \cong \mathbf{SepJoin}(\mathfrak{A}, 2) \cong F(\prod \mathfrak{A})/\sim \to 2 \cong \mathfrak{I}(F(\prod \mathfrak{A})/\sim)$. □

FiXme: Check below (especially posets vs dual posets) for errors.

COROLLARY 2047. $\mathsf{preStrd}$ is a complete lattice.

PROOF. Corollary 515. □

COROLLARY 2048. $\mathsf{preStrd}$ is a filtered filtrator.

PROOF. Theorem 531. □

FiXme: Try to prove that $\mathsf{preStrd}$ is atomic and moreover atomistic (under certain conditions). Other properties?