CHAPTER 5

Power of filters

1. Germs of functions

DEFINITION 2036. Functions $f, g \in \mathbf{Rel}(\operatorname{Ob}\mathcal{X}, B)$ *are of the same $\mathcal{X}$-germ* for a filter object $\mathcal{X}$ iff there exists $X \in \operatorname{up}\mathcal{X}$ such that $f|_X = g|_X$.

PROPOSITION 2037. Being of the same germ is an equivalence relation.

PROOF.
Reflexivity. Take arbitrary $X \in \operatorname{up}\mathcal{X}$.
Symmetry. Obvious.
Transitivity. Let $f|_X = g|_X$ and $g|_Y = h|_Y$. Then $f|_{X\cap Y} = h|_{X\cap Y}$.

□

DEFINITION 2038. A *germ* is an equivalence class of being the same germ.

OBVIOUS 2039. Every germ is a filter on **Set**.

THEOREM 2040. Let A, B be sets.
The following are mutually inverse bijections between monovalued reloids $f : A \to B$ with $\operatorname{dom} f = \mathcal{X}$ and $\mathcal{X}$-germs S of functions $A \to B$ for $\mathcal{X} \in \mathscr{F}A$:
1°. $f \mapsto \operatorname{up}^{\mathbf{Set}} f$;
2°. $S \mapsto s|_{\mathcal{X}}$ if $s \in S$.

The second bijection can also be written as $S \mapsto \left(\prod^{\mathsf{RLD}} S\right)|_{\mathcal{X}}$ or if $\operatorname{card} B \neq 1$ as $S \mapsto \prod^{\mathsf{RLD}} S$.

REMARK 2041. $s|_{\mathcal{X}}$ is always defined because S is nonempty (it is an equivalence class).

PROOF. First prove that $\operatorname{up}^{\mathbf{Set}} f$ is an $\mathcal{X}$-germ. Really, $F \in \operatorname{up}^{\mathbf{Set}} f \Leftrightarrow F \sqsupseteq f \Leftrightarrow F|_{\mathcal{X}} = f \Leftrightarrow \exists X \in \operatorname{up}\mathcal{X} : F|_X \sqsupseteq f$; thus $F, G \in \operatorname{up}^{\mathbf{Set}} f \Rightarrow \exists X \in \operatorname{up}\mathcal{X} : F|_X \sqsupseteq f \wedge \exists Y \in \operatorname{up}\mathcal{X} : G|_Y \sqsupseteq f \Rightarrow \exists X \in \operatorname{up}\mathcal{X} : F|_{X\cap Y} \sqsupseteq f \wedge \exists Y \in \operatorname{up}\mathcal{X} : G|_{X\cap Y} \sqsupseteq f \Rightarrow \exists Z \in \operatorname{up}\mathcal{X} : (F|_Z \sqsupseteq f \wedge G|_Z \sqsupseteq f) \Rightarrow \exists Z \in \operatorname{up}\mathcal{X} : (F \sqcap G)|_Z \sqsupseteq f$ and $F \in \operatorname{up}^{\mathbf{Set}} f \wedge \exists X \in \operatorname{up}\mathcal{X} : F|_X = G|_X \Rightarrow F \sqsupseteq f \wedge F|_{\mathcal{X}} = G|_{\mathcal{X}} \Rightarrow G|_{\mathcal{X}} \sqsupseteq f \Rightarrow G \in \operatorname{up}^{\mathbf{Set}} f$. We have proved that $\operatorname{up}^{\mathbf{Set}} f$ is an equivalence class of the suitable equivalence relation, that is $\operatorname{up}^{\mathbf{Set}} f$ is an $\mathcal{X}$-germ.

That $\prod^{\mathsf{RLD}} S$ is a monovalued reloid is obvious. Also $\operatorname{im} \prod^{\mathsf{RLD}} S = \mathcal{X}$ is obvious.

Now prove that our correspondences are mutually inverse.

Let $f_0 : A \to B$ be a monovalued reloid and $\operatorname{dom} f = \mathcal{X}$. Let $S = \operatorname{up}^{\mathbf{Set}} f_0$ and $f_1 = s|_{\mathcal{X}}$ for an $s \in S$. We need to prove $f_1 = f_0$. Really, $f_1 = F|_{\mathcal{X}}$ for an $F \in \operatorname{up}^{\mathbf{Set}} f_0$; thus $f_1 = f_0$.

Let S_0 be an $\mathcal{X}$-germ of functions $A \to B$. Let $f = s|_{\mathcal{X}}$ for an $s \in S_0$ and $S_1 = \operatorname{up}^{\mathbf{Set}} f$. We need to prove $S_1 = S_0$. Really,

$$S_1 = \operatorname{up}^{\mathbf{Set}}(s|_{\mathcal{X}}) = \left\{ \frac{F \in \mathbf{Set}}{F \sqsupseteq s|_{\mathcal{X}}} \right\} = \left\{ \frac{F \in \mathbf{Set}}{\exists X \in \operatorname{up}\mathcal{X} : F|_X \sqsupseteq s|_X} \right\} = \left\{ \frac{F \in \mathbf{Set}}{\exists X \in \operatorname{up}\mathcal{X} : F|_X = s|_X} \right\} = S_0.$$

$\left(\prod^{\mathsf{RLD}} S\right)|_{\mathcal{X}} = \prod^{\mathsf{RLD}}_{s\in S} s|_{\mathcal{X}} = s|_{\mathcal{X}}$ for every choice of $s \in S$.