CHAPTER 4

Categories of filters

In [1] two categories, whose objects are related with filters on sets, are defined and researched.

Accordingly [1] infinite product is defined just in the first (denoted $\mathscr{F}$ there) of these two categories. So we will for now consider the first category. (Usefulness of the second category for our research is questionable.)

Let $f : A \to B$ be a function, $\mathcal{A}$ be a filter on A.

PROPOSITION 2030. $\left\{ \frac{Y \in \mathscr{P} B}{\langle f^{-1} \rangle^* Y \in \mathcal{A}} \right\}$ is a filter.

PROOF. That it is an upper set is obvious.

Let $Y_0, Y_1 \in \left\{ \frac{Y \in \mathscr{P} B}{\langle f^{-1} \rangle^* Y \in \mathcal{A}} \right\}$. Then $\langle f^{-1} \rangle^* Y_0 \in \mathcal{A}$ and $\langle f^{-1} \rangle^* Y_1 \in \mathcal{A}$. We have

$$\langle f^{-1} \rangle^* (Y_0 \cap Y_1) = \langle f^{-1} \rangle^* Y_0 \cap \langle f^{-1} \rangle^* Y_1 \in \mathcal{A}$$

since f is monovalued. Thus $Y_0 \cap Y_1 \in \left\{ \frac{Y \in \mathscr{P} B}{\langle f^{-1} \rangle^* Y \in \mathcal{A}} \right\}$. □

THEOREM 2031. FiXme: Should be moved above in the book. $\left\{ \frac{Y \in \mathscr{P} B}{\langle f^{-1} \rangle^* Y \in \mathcal{A}} \right\}$ is equal to the filter generated by the filter base $\langle \langle f \rangle^* \rangle^* \mathcal{A}$, for every filter $\mathcal{A}$.

PROOF. Denote $\mathcal{B} = \left\{ \frac{Y \in \mathscr{P} B}{\langle f^{-1} \rangle^* Y \in \mathcal{A}} \right\}$, $\mathcal{C} = \langle \langle f \rangle^* \rangle^* \mathcal{A}$.

Let $Y \in \mathcal{C}$. Then $Y = \langle f \rangle^* A$ where $A \in \mathcal{A}$. Then $\langle f^{-1} \rangle^* \langle f \rangle^* A \supseteq A$ and so $\langle f^{-1} \rangle^* \langle f \rangle^* A \in \mathcal{A}$. This proves $\langle f \rangle^* A \in \mathcal{B}$, that is $Y \in \mathcal{B}$.

Let now $Y \in \mathcal{B}$. Then $\langle f \rangle^* \langle f^{-1} \rangle^* Y \subseteq Y$. Since $\langle f^{-1} \rangle^* Y \in \mathcal{A}$, we have that Y is a supset of some set of the form $\langle f \rangle^* A$, so $Y \in \mathcal{C}$. □

COROLLARY 2032. $\operatorname{up} \langle f \rangle \mathcal{A} = \left\{ \frac{Y \in \mathscr{P} B}{\langle f^{-1} \rangle^* Y \in \operatorname{up} \mathcal{A}} \right\}$.

DEFINITION 2033. The *category of filtered sets* **Filt** is the category defined as follows:

1°. Objects are pairs $(A, \mathcal{A})$ where A is a (small) set and $\mathcal{A}$ is a filter on A.
2°. Morphisms from $(A, \mathcal{A})$ to $(B, \mathcal{B})$ are functions $f : A \to B$ such that $\langle f \rangle \mathcal{A} \sqsubseteq \mathcal{B}$.
3°. Identities are identity functions.

To verify that it is a category is straightforward.

It is the same category as $\mathscr{F}$ in [1], as follows from an above proposition.

We will prove that starred reloidal product is a categorical product in this category. First we will prove the special case that binary reloidal product is a categorical product in this category.

THEOREM 2034. $\times^{\mathsf{RLD}}$ (together with projections $\Pr_0$ and $\Pr_1$) is a categorical product in **Filt**.

PROOF. Let our objects be $\mathcal{A}$, $\mathcal{B}$.

Denote p the left projection from $\operatorname{Base}(\mathcal{A}) \times \operatorname{Base}(\mathcal{B})$ to $\operatorname{Base}(\mathcal{A})$.