CHAPTER 3

Equalizers and co-Equalizers in Certain Categories

It is a rough draft. Errors are possible.

FiXme: Change notation $\prod \to \prod^{(L)}$.

1. Equalizers

Categories $\operatorname{cont}(\mathcal{C})$ are defined above.

I will denote W the forgetful functor from $\operatorname{cont}(\mathcal{C})$ to $\mathcal{C}$.

In the definition of the category $\operatorname{cont}(\mathcal{C})$ take values of $\uparrow$ as principal morphisms. FiXme: Wording.

LEMMA 2018. Let $f : X \to Y$ be a morphism of the category $\operatorname{cont}(\mathcal{C})$ where $\mathcal{C}$ is a concrete category (so $Wf = \uparrow \varphi$ for a **Rel**-morphism φ because f is principal) and $\operatorname{im} \varphi = A \subseteq \operatorname{Ob} Y$. Factor it $\varphi = \mathcal{E}^{\operatorname{Ob} Y} \circ u$ where $u : \operatorname{Ob} X \to A$ using properties of **Set**. Then u is a morphism of $\operatorname{cont}(\mathcal{C})$ (that is a continuous function $X \to \iota_A Y$).

PROOF. $(\mathcal{E}^{\operatorname{Ob} Y})^{-1} \circ \varphi = (\mathcal{E}^{\operatorname{Ob} Y})^{-1} \circ \mathcal{E}^{\operatorname{Ob} Y} \circ u$;

$(\mathcal{E}_{\mathcal{C}}^{\operatorname{Ob} Y})^{-1} \circ \uparrow \varphi = (\mathcal{E}_{\mathcal{C}}^{\operatorname{Ob} Y})^{-1} \circ \mathcal{E}_{\mathcal{C}}^{\operatorname{Ob} Y} \circ \uparrow u$;

$(\mathcal{E}_{\mathcal{C}}^{\operatorname{Ob} Y})^{-1} \circ \uparrow \varphi = \uparrow u$;

$X \sqsubseteq (\uparrow u)^{-1} \circ \pi_A Y \circ \uparrow u \Leftrightarrow X \sqsubseteq (\uparrow \varphi)^{-1} \circ \mathcal{E}_{\mathcal{C}}^{\operatorname{Ob} Y} \circ \pi_A Y \circ (\mathcal{E}_{\mathcal{C}}^{\operatorname{Ob} Y})^{-1} \circ \uparrow \varphi \Leftrightarrow X \sqsubseteq (\uparrow \varphi)^{-1} \circ \mathcal{E}_{\mathcal{C}}^{\operatorname{Ob} Y} \circ (\mathcal{E}_{\mathcal{C}}^{\operatorname{Ob} Y})^{-1} \circ Y \circ \mathcal{E}_{\mathcal{C}}^{\operatorname{Ob} Y} \circ (\mathcal{E}_{\mathcal{C}}^{\operatorname{Ob} Y})^{-1} \circ \uparrow \varphi \Leftrightarrow X \sqsubseteq (\uparrow \varphi)^{-1} \circ Y \circ \uparrow \varphi \Leftrightarrow X \sqsubseteq (Wf)^{-1} \circ Y \circ Wf$ what is true by definition of continuity. □

Equational definition of equalizers:

http://nforum.mathforge.org/comments.php?DiscussionID=5328/

THEOREM 2019. The following is an equalizer of parallel morphisms $f, g : A \to B$ of category $\operatorname{cont}(\mathcal{C})$:

- the object $X = \iota_{\left\{ \frac{x \in \operatorname{Ob} A}{fx = gx} \right\}} A$;
- the morphism $\mathcal{E}^{\operatorname{Ob} X, \operatorname{Ob} A}$ considered as a morphism $X \to A$.

PROOF. Denote $e = \mathcal{E}^{\operatorname{Ob} X, \operatorname{Ob} A}$.

Let $f \circ z = g \circ z$ for some morphism z.

Let's prove $e \circ u = z$ for some $u : \operatorname{Src} z \to X$. Really, as a morphism of **Set** it exists and is unique.

Consider z as as a generalized element.

$f(z) = g(z)$. So $z \in X$ (that is $\operatorname{Dst} z \in X$). Thus $z = e \circ u$ for some u (by properties of **Set**). The generalized element u is a $\operatorname{cont}(\mathcal{C})$-morphism because of the lemma above. It is unique by properties of **Set**. □

We can (over)simplify the above theorem by the obvious below:

OBVIOUS 2020. $\left\{ \frac{x \in \operatorname{Ob} A}{fx = gx} \right\} = \operatorname{dom}(f \cap g)$.

2. Co-equalizers

http://math.stackexchange.com/questions/539717/how-to-construct-co-equalizers-in-mathbftop

Let $\sim$ be an equivalence relation. Let's denote π its canonical projection.