APPENDIX A

Using logic of generalizations

A.1. Logic of generalization

In mathematics it is often encountered that a smaller set S naturally bijectively corresponds to a subset R of a larger set B. (In other words, there is specified an injection from S to B.) It is a widespread practice to equate S with R.

Remark 1946. I denote the first set S from the first letter of the word "small" and the second set B from the first letter of the word "big", because S is intuitively considered as smaller than B. (However we do not require $\operatorname{card} S < \operatorname{card} B$.)

The set B is considered as a generalization of the set S, for example: whole numbers generalizing natural numbers, rational numbers generalizing whole numbers, real numbers generalizing rational numbers, complex numbers generalizing real numbers, etc.

But strictly speaking this equating may contradict to the axioms of ZF/ZFC because we are not insured against $S \cap B \neq \emptyset$ incidents. Not wonderful, as it is often labeled as "without proof".

To work around of this (and formulate things exactly what could benefit computer proof assistants) we will replace the set B with a new set B' having a bijection $M : B \to B'$ such that $S \subseteq B'$. (I call this bijection M from the first letter of the word "move" which signifies the move from the old set B to a new set B').

The following is a formal but rather silly formalization of this situation in ZF. (A more natural formalization may be done in a smarter formalistic, such as dependent type theory.)

A.1.1. The formalistic. Let S and B be sets. Let E be an injection from S to B. Let $R = \operatorname{im} E$.

Let $t = \mathscr{P} \bigcup\bigcup S$.

Let $M(x) = \begin{cases} E^{-1}x & \text{if } x \in R; \\ (t, x) & \text{if } x \notin R. \end{cases}$

Recall that in standard ZF $(t, x) = \{t, \{t, x\}\}$ by definition.

Theorem 1947. $(t, x) \notin S$.

Proof. Suppose $(t, x) \in S$. Then $\{t, \{t, x\}\} \in S$. Consequently $\{t\} \in \bigcup S$; $\{t\} \subseteq \bigcup\bigcup S$; $\{t\} \in \mathscr{P} \bigcup\bigcup S$; $\{t\} \in t$ what contradicts to the axiom of foundation (aka axiom of regularity). □

Definition 1948. Let $B' = \operatorname{im} M$.

Theorem 1949. $S \subseteq B'$.

Proof. Let $x \in S$. Then $Ex \in R$; $M(Ex) = E^{-1}Ex = x$; $x \in \operatorname{im} M = B'$. □

Obvious 1950. E is a bijection from S to R.

Theorem 1951. M is a bijection from B to B'.

Proof. Surjectivity of M is obvious. Let's prove injectivity. Let $a, b \in B$ and $M(a) = M(b)$. Consider all cases: □