PROOF.

$$a_0 \times^{\mathsf{RLD}} b_0 \left[f \times^{(A)} g \right] a_1 \times^{\mathsf{RLD}} b_1 \Leftrightarrow$$
$$\forall A_0 \in a_0, B_0 \in b_0, A_1 \in a_1, B_1 \in b_1 : A_0 \times B_0 \left[f \times^{(A)} g \right]^* A_1 \times B_1.$$

$$A_0 \times B_0 \left[f \times^{(A)} g \right]^* A_1 \times B_1 \Leftrightarrow A_0 \times B_0 \left[f \times^{(C)} g \right]^* A_1 \times B_1 \Leftrightarrow A_0 \left[f \right]^* A_1 \wedge B_0 \left[g \right]^* B_1.$$

(Here by $A_0 \times B_0 \left[f \times^{(C)} g \right]^* A_1 \times B_1$ I mean $\uparrow^{\mathsf{FCD}(\operatorname{Base} a, \operatorname{Base} b)} (A_0 \times B_0) \left[f \times^{(C)} g \right]^* \uparrow^{\mathsf{FCD}(\operatorname{Base} a, \operatorname{Base} b)} (A_1 \times B_1)$.)

Thus it is equivalent to $a_0 \left[f \right] a_1 \wedge b_0 \left[g \right] b_1$ that is $a_0 \times^{\mathsf{FCD}} b_0 \left[f \times^{(C)} g \right]^* a_1 \times^{\mathsf{FCD}} b_1$.

(It was used the corollary 1626.) □

Can the above theorem be generalized for the infinitary case?

21.15. Cross-inner and cross-outer product

Let f be an indexed family of funcoids.

DEFINITION 1818. $\prod^{\mathrm{in}}_{i \in \operatorname{dom} f} f = \prod^{(C)}_{i \in \operatorname{dom} f} (\mathsf{RLD})_{\mathrm{in}} f_i$ (*cross-inner product*).

DEFINITION 1819. $\prod^{\mathrm{out}}_{i \in \operatorname{dom} f} f = \prod^{(C)}_{i \in \operatorname{dom} f} (\mathsf{RLD})_{\mathrm{out}} f_i$ (*cross-outer product*).

PROPOSITION 1820. $\prod^{\mathrm{in}}_{i \in \operatorname{dom} f} f$ and $\prod^{\mathrm{out}}_{i \in \operatorname{dom} f} f$ are funcoids (not just pointfree funcoids).

PROOF. They are both morphisms $\operatorname{StarHom}(\lambda i \in \operatorname{dom} f : \operatorname{Src} f_i) \to \operatorname{StarHom}(\lambda i \in \operatorname{dom} f : \operatorname{Src} f_i)$ for the category of multireloids with star-morphisms, that is $\operatorname{StarHom}(\lambda i \in \operatorname{dom} f : \operatorname{Src} f_i)$ is the set of filters on the cartesian product $\prod_{i \in \operatorname{dom} f} \operatorname{Src} f_i$ and likewise for $\operatorname{StarHom}(\lambda i \in \operatorname{dom} f : \operatorname{Src} f_i)$. □

OBVIOUS 1821. For every funcoids f and g

1°. $f \times^{\mathrm{in}} g = (\mathsf{RLD})_{\mathrm{in}} f \times^{(C)} (\mathsf{RLD})_{\mathrm{in}} g$;
2°. $f \times^{\mathrm{out}} g = (\mathsf{RLD})_{\mathrm{out}} f \times^{(C)} (\mathsf{RLD})_{\mathrm{out}} g$.

COROLLARY 1822.

1°. $\langle f \times^{\mathrm{in}} g \rangle a = (\mathsf{RLD})_{\mathrm{in}} g \circ a \circ (\mathsf{RLD})_{\mathrm{in}} f^{-1}$;
2°. $\langle f \times^{\mathrm{out}} g \rangle a = (\mathsf{RLD})_{\mathrm{out}} g \circ a \circ (\mathsf{RLD})_{\mathrm{out}} f^{-1}$

COROLLARY 1823. For every funcoids f and g and filters a and b on suitable sets:

1°. $a \left[f \times^{\mathrm{in}} g \right] b \Leftrightarrow b \not\asymp (\mathsf{RLD})_{\mathrm{in}} g \circ a \circ (\mathsf{RLD})_{\mathrm{in}} f^{-1} \Leftrightarrow b \circ (\mathsf{RLD})_{\mathrm{in}} f \not\asymp (\mathsf{RLD})_{\mathrm{in}} g \circ a$;
2°. $a \left[f \times^{\mathrm{out}} g \right] b \Leftrightarrow b \not\asymp (\mathsf{RLD})_{\mathrm{out}} g \circ a \circ (\mathsf{RLD})_{\mathrm{out}} f^{-1} \Leftrightarrow b \circ (\mathsf{RLD})_{\mathrm{out}} f \not\asymp (\mathsf{RLD})_{\mathrm{out}} g \circ a$.

PROPOSITION 1824. Knowing that every f_i is nonzero, we can restore the values of f_i from the value of $\prod^{\mathrm{in}}_{i \in \operatorname{dom} f} f$.

PROOF. It follows that every $(\mathsf{RLD})_{\mathrm{in}} f_i$ is nonzero, thus we can restore each $(\mathsf{RLD})_{\mathrm{in}} f_i$ from $\prod^{(C)}_{i \in \operatorname{dom} f} (\mathsf{RLD})_{\mathrm{in}} f_i = \prod^{\mathrm{in}}_{i \in \operatorname{dom} f} f$ and then we know $f_i = (\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f_i$. □

EXAMPLE 1825. The values of f and g cannot be restored from $f \times^{\mathrm{out}} g$ for some nonzero funcoids f and g.