21.10.5. Cross-composition product of funcoids. Let a be a an anchored relation of the form $\mathfrak{A}$ and $\operatorname{dom}\mathfrak{A} = n$.

Let every f_i (for all $i \in n$) be a pointfree funcoid with $\operatorname{Src} f_i = \mathfrak{A}_i$.

The star-composition of a with f is an anchored relation of the form $\lambda i \in \operatorname{dom}\mathfrak{A} : \operatorname{Dst} f_i$ defined by the formula

$$L \in \operatorname{GR}\operatorname{StarComp}(a, f) \Leftrightarrow (\lambda i \in n : \langle f_i^{-1}\rangle L_i) \in \operatorname{GR} a.$$

THEOREM 1736. Let $\operatorname{Src} f_i$ be separable starrish join-semilattice and $\operatorname{Dst} f_i$ be a starrish join-semilattice for every $i \in n$ for a set n. Let $\operatorname{form} a = \prod_{i\in n}(\operatorname{Src} f_i)$.

1°. If a is a prestaroid then $\operatorname{StarComp}(a, f)$ is a prestaroid.
2°. If a is a staroid and $\operatorname{Src} f_i$ are strongly separable then $\operatorname{StarComp}(a, f)$ is a staroid.
3°. If a is a completary staroid and then $\operatorname{StarComp}(a, f)$ is a completary staroid.

PROOF. We have $\langle f_i^{-1}\rangle(X \sqcup Y) = \langle f_i^{-1}\rangle X \sqcup \langle f_i^{-1}\rangle Y$ by theorem 1498.

1°. Let $L \in \prod_{i\in(\operatorname{arity} f)\setminus\{k\}}(\operatorname{form} f_i)$ for some $k \in n$ and $X, Y \in \operatorname{form} f_k$. Then

$$\begin{aligned}
&X \sqcup Y \in \langle \operatorname{StarComp}(a, f)\rangle_k^* L \Leftrightarrow\\
&\left(\lambda i \in \operatorname{dom} f : \langle f_i^{-1}\rangle\left(\begin{cases} X \sqcup Y & \text{if } i = k\\ L_i & \text{if } i \neq k\end{cases}\right)_i\right) \in \operatorname{GR} a \Leftrightarrow\\
&\left(\lambda i \in \operatorname{dom} f : \left(\begin{cases} \langle f_i^{-1}\rangle X \sqcup \langle f_i^{-1}\rangle Y & \text{if } i = k\\ \langle f_i^{-1}\rangle L_i & \text{if } i \neq k\end{cases}\right)_i\right) \in \operatorname{GR} a \Leftrightarrow\\
&\langle f_i^{-1}\rangle X \sqcup \langle f_i^{-1}\rangle Y \in \langle a\rangle_k^*(\lambda i \in (\operatorname{dom} f)\setminus\{k\} : \langle f_i^{-1}\rangle L_i) \Leftrightarrow\\
&\langle f_i^{-1}\rangle X \in \langle a\rangle_k^*(\lambda i \in n\setminus\{k\} : \langle f_i^{-1}\rangle L_i) \vee \langle f_i^{-1}\rangle Y \in \langle a\rangle_k^*(\lambda i \in n\setminus\{k\} : \langle f_i^{-1}\rangle L_i) \Leftrightarrow\\
&\left(\lambda i \in \operatorname{dom} f : \left(\begin{cases} \langle f_i^{-1}\rangle X & \text{if } i = k\\ \langle f_i^{-1}\rangle L_i & \text{if } i \neq k\end{cases}\right)_i\right) \in \operatorname{GR} a \vee\\
&\left(\lambda i \in \operatorname{dom} f : \left(\begin{cases} \langle f_i^{-1}\rangle Y & \text{if } i = k\\ \langle f_i^{-1}\rangle L_i & \text{if } i \neq k\end{cases}\right)_i\right) \in \operatorname{GR} a \Leftrightarrow\\
&\left(\lambda i \in \operatorname{dom} f : \langle f_i^{-1}\rangle\left(\begin{cases} X & \text{if } i = k\\ L_i & \text{if } i \neq k\end{cases}\right)_i\right) \in \operatorname{GR} a \vee\\
&\left(\lambda i \in \operatorname{dom} f : \langle f_i^{-1}\rangle\left(\begin{cases} Y & \text{if } i = k\\ L_i & \text{if } i \neq k\end{cases}\right)_i\right) \in \operatorname{GR} a \Leftrightarrow\\
&X \in \langle \operatorname{StarComp}(a, f)\rangle_k^* L \vee Y \in \langle \operatorname{StarComp}(a, f)\rangle_k^* L.
\end{aligned}$$

Thus $\operatorname{StarComp}(a, f)$ is a pre-staroid.

2°. $\langle f_i^{-1}\rangle$ are monotone functions by the proposition 1497. Thus $\langle f_i^{-1}\rangle X_i \sqsubseteq \langle f_i^{-1}\rangle Y_i$ if $X, Y \in \prod_{i\in(\operatorname{arity} f)\setminus\{k\}}(\operatorname{form} f_i)$ and $X \sqsubseteq Y$. So if a is a staroid and $X \in \operatorname{GR}\operatorname{StarComp}(a, f)$ then $(\lambda i \in \operatorname{dom} f : \langle f_i^{-1}\rangle X_i) \in \operatorname{GR} a$ then $(\lambda i \in \operatorname{dom} f : \langle f_i^{-1}\rangle Y_i) \in \operatorname{GR} a$ that is $Y \in \operatorname{GR}\operatorname{StarComp}(a, f)$.

3°.

$$\begin{aligned}
&L_0 \sqcup L_1 \in \operatorname{GR}\operatorname{StarComp}(a, f) \Leftrightarrow\\
&(\lambda i \in n : \langle f_i^{-1}\rangle(L_0 \sqcup L_1)i) \in \operatorname{GR} a \Leftrightarrow\\
&(\lambda i \in n : \langle f_i^{-1}\rangle L_0 i \sqcup \langle f_i^{-1}\rangle L_1 i) \in \operatorname{GR} a \Leftrightarrow\\
&\exists c \in \{0, 1\} : (\lambda i \in n : \langle f_i^{-1}\rangle L_{c(i)} i) \in \operatorname{GR} a \Leftrightarrow\\
&\exists c \in \{0, 1\} : (\lambda i \in n : L_{c(i)} i) \in \operatorname{GR}\operatorname{StarComp}(a, f).
\end{aligned}$$

□