PROOF.

$$\begin{gathered} L \in \left\langle \prod^{(C)} f \right\rangle \prod a \Leftrightarrow \\ L \in \operatorname{StarComp}\left(\prod a, f\right) \Leftrightarrow \\ \exists y \in \prod a \forall i \in n : y_i \; f_i \; L_i \Leftrightarrow \\ \exists y \in \prod a \forall i \in n : \{y_i\} \not\asymp \left\langle f_i^{-1} \right\rangle^* \{L_i\} \Leftrightarrow \\ \forall i \in n \exists y \in a_i : \{y\} \not\asymp \left\langle f_i^{-1} \right\rangle^* \{L_i\} \Leftrightarrow \\ \forall i \in n : a_i \not\asymp \left\langle f_i^{-1} \right\rangle^* \{L_i\} \Leftrightarrow \\ \forall i \in n : \{L_i\} \not\asymp \langle f_i \rangle^* a_i \Leftrightarrow \\ \forall i \in n : L_i \in \langle f_i \rangle^* a_i \Leftrightarrow \\ L \in \prod_{i \in n} \langle f_i \rangle^* a_i. \end{gathered}$$

□

21.10.4. Star composition of Rel-morphisms. Define *star composition* for an n-ary anchored relation a and an n-indexed family f of **Rel**-morphisms as an n-ary anchored relation complying with the formulas:

$$\begin{gathered} \operatorname{Obj}_{\operatorname{StarComp}(a,f)} = \lambda i \in \operatorname{arity} a : \operatorname{Dst} f_i; \\ \operatorname{arity} \operatorname{StarComp}(a,f) = \operatorname{arity} a; \\ L \in \operatorname{GR} \operatorname{StarComp}(a,f) \Leftrightarrow L \in \operatorname{StarComp}(\operatorname{GR} a, \operatorname{GR} \circ f). \end{gathered}$$

(Here I denote $\operatorname{GR}(A,B,f) = f$ for every **Rel**-morphism f.)

PROPOSITION 1732.

$$b \not\asymp \operatorname{StarComp}(a,f) \Leftrightarrow \exists x \in a, y \in b \forall j \in n : x_j \; \operatorname{GR}(f_j) \; y_j.$$

PROOF. From the previous section. □

THEOREM 1733. Relations with above defined compositions form a quasi-invertible category with star-morphisms.

PROOF. We need to prove:

1°. $\operatorname{StarComp}(\operatorname{StarComp}(m,f),g) = \operatorname{StarComp}(m, \lambda i \in \operatorname{arity} m : g_i \circ f_i)$;
2°. $\operatorname{StarComp}(m, \lambda i \in \operatorname{arity} m : 1_{\operatorname{Obj}_m i}) = m$;
3°. $b \not\asymp \operatorname{StarComp}(a,f) \Leftrightarrow a \not\asymp \operatorname{StarComp}(b, f^\dagger)$

(the rest is obvious).

It follows from the previous section. □

PROPOSITION 1734. $\operatorname{StarComp}(a \sqcup b, f) = \operatorname{StarComp}(a,f) \sqcup \operatorname{StarComp}(b,f)$ for an n-ary anchored relations a, b and an n-indexed family f of **Rel**-morphisms.

PROOF. It follows from the previous section. □

THEOREM 1735. Cross-composition product of a family of **Rel**-morphisms is a principal funcoid.

PROOF. By the proposition and symmetry $\prod^{(C)} f$ is a pointfree funcoid. Obviously it is a funcoid $\prod_{i \in n} \operatorname{Src} f_i \to \prod_{i \in n} \operatorname{Dst} f_i$. Its completeness (and dually co-completeness) is obvious. □