The reverse implication is obvious. Let $\mathcal{L} \cup \{(i, \mathcal{X} \sqcup \mathcal{Y})\} \in \mathrm{GR} \upuparrows f$. Then for every $L \in \mathrm{up}\,\mathcal{L}$ and $X \in \mathrm{up}\,\mathcal{X}$, $Y \in \mathrm{up}\,\mathcal{Y}$ we have $L \cup \{(i, X \sqcup^{\mathfrak{Z}_i} Y)\} \in \mathrm{GR}\, f$ and thus

$$L \cup \{(i, X)\} \in \mathrm{GR}\, f \vee L \cup \{(i, Y)\} \in \mathrm{GR}\, f$$

consequently $\mathcal{L} \cup \{(i, \mathcal{X})\} \in \mathrm{GR} \upuparrows f \vee \mathcal{L} \cup \{(i, \mathcal{Y})\} \in \mathrm{GR} \upuparrows f$.

It is left to prove that $\upuparrows f$ is an upper set, but this is obvious. □

There is a conjecture similar to the above theorems:

CONJECTURE 1676. $L \in \upuparrows [f]^* \Rightarrow \upuparrows [f]^* \cap \prod_{i\in \mathrm{dom}\,\mathfrak{A}} \mathrm{atoms}\, L_i \neq \emptyset$ for every multifuncoid f for the filtrator $(\mathscr{F}^n, \mathfrak{Z}^n)$.

CONJECTURE 1677. Let $(\mathfrak{A}, \mathfrak{Z})$ be a powerset filtrator, let n be an index set. Consider the filtrator $(\mathscr{F}^n, \mathfrak{Z}^n)$. Then if f is a completary staroid of the form $\mathfrak{Z}^n$, then $\upuparrows f$ is a completary staroid of the form $\mathfrak{A}^n$.

EXAMPLE 1678. There is such an anchored relation f that for some $k \in \mathrm{dom}\, f$

$$\langle \upuparrows\uparrow f\rangle^*{}_k \mathcal{L} \neq \bigsqcup_{a\in\prod_{i\in(\mathrm{dom}\, f)\setminus\{k\}} \mathrm{atoms}\,\mathcal{L}_i} \langle \upuparrows\uparrow f\rangle^*_k a.$$

PROOF. Take $\mathcal{P} \in \mathrm{GR}\, f$ from the counter-example 1661. We have

$$\forall a \in \prod_{i\in \mathrm{dom}\, f} \mathrm{atoms}\,\mathcal{P}_i : a \notin \mathrm{GR}\,\mathcal{P}.$$

Take $k = 1$.

Let $\mathcal{L} = \mathcal{P}|_{(\mathrm{dom}\, f)\setminus\{k\}}$. Then $a \notin \mathrm{GR} \upuparrows\uparrow f$ and thus $a_k \asymp \langle \upuparrows\uparrow f\rangle^*{}_k a|_{(\mathrm{dom}\, f)\setminus\{k\}}$.

Consequently $\mathcal{P}_k \asymp \langle \upuparrows\uparrow f\rangle^*{}_k a|_{(\mathrm{dom}\, f)\setminus\{k\}}$ and thus $\mathcal{P}_k \asymp \bigsqcup_{a\in\prod_{i\in(\mathrm{dom}\, f)\setminus\{k\}} \mathrm{atoms}\,\mathcal{L}_i} \langle \upuparrows\uparrow f\rangle^*{}_k a$ because $\mathcal{P}_k$ is principal.

But $\mathcal{P}_k \not\asymp \langle \upuparrows\uparrow f\rangle^*{}_k \mathcal{L}$. Thus follows $\langle \upuparrows\uparrow f\rangle^*_k \mathcal{L} \neq \bigsqcup_{a\in\prod_{i\in(\mathrm{dom}\, f)\setminus\{k\}} \mathrm{atoms}\,\mathcal{L}_i} \langle \upuparrows\uparrow f\rangle^*{}_k a$. □

21.6. Join of multifuncoids

Mults are ordered by the formula $f \sqsubseteq g \Leftrightarrow \langle f\rangle^* \sqsubseteq \langle g\rangle^*$ where $\sqsubseteq$ in the right part of this formula is the product order. I will denote $\sqcap$, $\sqcup$, $\prod$, $\bigsqcup$ (without an index) the order poset operations on the poset of mults.

REMARK 1679. To describe this, the definition of product order is used twice. Let f and g be mults of the same form $(\mathfrak{A}, \mathfrak{Z})$

$$\langle f\rangle^* \sqsubseteq \langle g\rangle^* \Leftrightarrow \forall i \in \mathrm{dom}\,\mathfrak{Z} : \langle f\rangle^*_i \sqsubseteq \langle g\rangle^*_i;$$

$$\langle f\rangle^*_i \sqsubseteq \langle g\rangle^*_i \Leftrightarrow \forall L \in \prod \mathfrak{Z}|_{(\mathrm{dom}\,\mathfrak{Z})\setminus\{i\}} : \langle f\rangle^*_i L \sqsubseteq \langle g\rangle^*_i L.$$

OBVIOUS 1680. $(\bigsqcup F)K = \bigsqcup_{f\in F} fK$ for every set F of mults of the same form $\mathfrak{Z}$ and $K \in \prod \mathfrak{Z}$ whenever every $\bigsqcup_{f\in F} fK$ is defined.

THEOREM 1681. $f \sqcup^{\mathsf{pFCD}(\mathfrak{A})} g = f \sqcup g$ for every multifuncoids f and g for the same indexed family of starrish join-semilattices filtrators.

PROOF. $\alpha_i x \overset{\mathrm{def}}{=} \langle f_i\rangle^* x \sqcup \langle g_i\rangle^* x$. It is enough to prove that α is a multifuncoid. We need to prove:

$$L_i \not\asymp \alpha_i L|_{(\mathrm{dom}\, L)\setminus\{i\}} \Leftrightarrow L_j \not\asymp \alpha_j L|_{(\mathrm{dom}\, L)\setminus\{j\}}.$$