CONJECTURE 1662. Filtrators of staroids on powersets are join-closed.

21.5. Multifuncoids

DEFINITION 1663. Let $(\mathfrak{A}_i, \mathfrak{Z}_i)$ (where $i \in n$ for an index set n) be an indexed family of filtrators.

I call a *mult* f of the form $(\mathfrak{A}_i, \mathfrak{Z}_i)$ the triple $f = (\operatorname{base} f, \operatorname{core} f, \langle f \rangle^*)$ of n-indexed families of posets $\operatorname{base} f$ and $\operatorname{core} f$ and $\langle f \rangle^*$ of functions where for every $i \in n$

$$\langle f \rangle^*_i : \prod (\operatorname{core} f)_i|_{(\operatorname{dom} \mathfrak{A}) \setminus \{i\}} \to (\operatorname{base} f)_i.$$

I call $(\operatorname{base} f, \operatorname{core} f)$ the *form* of the mult f.

REMARK 1664. I call it *mult* because it comprises multiple functions $\langle f \rangle^*_i$.

DEFINITION 1665. A mult *on powersets* is a mult such that every $((\operatorname{base} f)_i, (\operatorname{core} f)_i)$ is a powerset filtrator.

DEFINITION 1666. I will call a *relational mult* a mult f such that every $(\operatorname{base} f)_i$ is a set and for every $i, j \in n$ and $L \in \prod \operatorname{core} f$

$$L_i \in \langle f \rangle^*_i L|_{(\operatorname{dom} L) \setminus \{i\}} \Leftrightarrow L_j \in \langle f \rangle^*_j L|_{(\operatorname{dom} L) \setminus \{j\}}.$$

I denote $\operatorname{arity} f = n$.

DEFINITION 1667. *Prestaroidal mult* is a relational mult of the form $(\mathfrak{A}, \lambda i \in \operatorname{dom} \mathfrak{A} : \mathfrak{S}(\mathfrak{A}_i))$ (where $\mathfrak{A}$ is a poset), that is such that $\langle f \rangle^*_i L$ is a free star for every $i \in n$ and $L \in \prod_{i \in (\operatorname{dom} L) \setminus \{i\}} \operatorname{core} f_i$.

DEFINITION 1668. I will call a *multifuncoid* a mult f such that $(\operatorname{core} f)_i \subseteq (\operatorname{base} f)_i$ (thus having a filtrator $((\operatorname{base} f)_i, (\operatorname{core} f)_i)$) for each $i \in n$ and for every $i, j \in n$ and $L \in \prod \operatorname{core} f$

$$L_i \not\asymp \langle f \rangle^*_i L|_{(\operatorname{dom} L) \setminus \{i\}} \Leftrightarrow L_j \not\asymp \langle f \rangle^*_j L|_{(\operatorname{dom} L) \setminus \{j\}}. \tag{33}$$

I denote the set of multifuncoids for a family $(\mathfrak{A}, \mathfrak{Z})$ of filtrators as $\mathsf{pFCD}(\mathfrak{A}, \mathfrak{Z})$ or just $\mathsf{pFCD}(\mathfrak{A})$ when $\mathfrak{Z}$ is clear from context.

DEFINITION 1669. To every multifuncoid f corresponds an anchored relation g by the formula (with arbitrary $i \in \operatorname{arity} f$)

$$L \in \operatorname{GR} g \Leftrightarrow L_i \not\asymp \langle f \rangle^*_i L|_{(\operatorname{dom} L) \setminus \{i\}}.$$

PROPOSITION 1670. Prestaroidal mults $\Lambda g = f$ of the form $(\mathfrak{Z}, \lambda i \in \operatorname{dom} \mathfrak{Z} : \mathfrak{S}(\mathfrak{Z}_i))$ bijectively correspond to pre-staroids g of the form $\mathfrak{Z}$ by the formulas (for every $K \in \prod \mathfrak{Z}$, $i \in \operatorname{dom} \mathfrak{Z}$, $L \in \prod_{j \in (\operatorname{dom} \mathfrak{A}) \setminus \{i\}} \mathfrak{Z}_j$, $X \in \mathfrak{Z}_i$)

$$K \in \operatorname{GR} g \Leftrightarrow K_i \in \langle f \rangle^*_i K|_{(\operatorname{dom} L) \setminus \{i\}}; \tag{34}$$

$$X \in \langle f \rangle^*_i L \Leftrightarrow L \cup \{(i, X)\} \in \operatorname{GR} g. \tag{35}$$

PROOF. If f is a prestaroidal mult, then obviously formula (34) defines an anchored relation between posets. $(\operatorname{val} g)_i = \langle f \rangle^*_i L$ is a free star. Thus g is a prestaroid.

If g is a prestaroid, then obviously formula (35) defines a relational mult. This mult is obviously prestaroidal.

It remains to prove that these correspondences are inverse of each other.

Let f_0 be a prestaroidal mult, g be the pre-staroid corresponding to f by formula (34), and f_1 be the prestaroidal mult corresponding to g by formula (35). Let's prove $f_0 = f_1$. Really,

$$X \in \langle f_1 \rangle^*_i L \Leftrightarrow L \cup \{(i, X)\} \in \operatorname{GR} g \Leftrightarrow X \in \langle f_0 \rangle^*_i L.$$