PROPOSITION 1640. An anchored relation f between posets whose form is a family of join-semilattices is a completary staroid iff both:

1°. $L_0 \sqcup L_1 \in \operatorname{GR} f \Leftrightarrow \exists c \in \{0,1\}^n : (\lambda i \in n : L_{c(i)} i) \in \operatorname{GR} f$ for every $L_0, L_1 \in \prod \operatorname{form} f$.

2°. If $L \in \prod \operatorname{form} f$ and $L_i = \bot^{(\operatorname{form} f)_i}$ for some $i \in \operatorname{arity} f$ then $L \notin \operatorname{GR} f$.

PROOF. Let the formulas 1° and 2° hold. Then f is an upper set: Let $L_0 \sqsubseteq L_1$ for some $L_0, L_1 \in \prod \operatorname{form} f$ and $L_0 \in f$. Then taking $c = n \times \{0\}$ we get $\lambda i \in n : L_{c(i)} i = \lambda i \in n : L_0 i = L_0 \in \operatorname{GR} f$ and thus $L_1 = L_0 \sqcup L_1 \in \operatorname{GR} f$.

Thus to finish the proof it is enough to show that

$$L_0 \sqcup L_1 \in \operatorname{GR} f \Leftrightarrow \forall K \in \prod \operatorname{form} f : (K \sqsupseteq L_0 \wedge K \sqsupseteq L_1 \Rightarrow K \in \operatorname{GR} f)$$

under condition that $\operatorname{GR} f$ is an upper set. But this equivalence is obvious in both directions. □

PROPOSITION 1641. Every completary staroid is a staroid.

PROOF. Let f be a completary staroid.

Let $i \in \operatorname{arity} f$, $K \in \prod_{i \in (\operatorname{arity} f) \setminus \{i\}} (\operatorname{form} f)_i$. Let $L_0 = K \cup \{(i, X_0)\}$, $L_1 = K \cup \{(i, X_1)\}$ for some $X_0, X_1 \in \mathfrak{A}_i$.

Let

$$\forall Z \in \mathfrak{A}_i : (Z \sqsupseteq X_0 \wedge Z \sqsupseteq X_1 \Rightarrow Z \in (\operatorname{val} f)_i K);$$

then

$$\forall Z \in \mathfrak{A}_i : (Z \sqsupseteq X_0 \wedge Z \sqsupseteq X_1 \Rightarrow K \cup \{(i, Z)\} \in \operatorname{GR} f).$$

If $z \sqsupseteq L_0 \wedge z \sqsupseteq L_1$ then $z \sqsupseteq K \cup \{(i, z_i)\}$, thus taking into account that $\operatorname{GR} f$ is an upper set,

$$\forall z \in \prod \mathfrak{A} : (z \sqsupseteq L_0 \wedge z \sqsupseteq L_1 \Rightarrow K \cup \{(i, z_i)\} \in \operatorname{GR} f).$$
$$\forall z \in \prod \mathfrak{A} : (z \sqsupseteq L_0 \wedge z \sqsupseteq L_1 \Rightarrow z \in \operatorname{GR} f).$$

Thus, by the definition of completary staroid, $L_0 \in \operatorname{GR} f \vee L_1 \in \operatorname{GR} f$ that is

$$X_0 \in (\operatorname{val} f)_i K \vee X_1 \in (\operatorname{val} f)_i K.$$

So $(\operatorname{val} f)_i K$ is a free star (taken into account that $z_i = \bot^{(\operatorname{form} f)_i} \Rightarrow z \notin \operatorname{GR} f$ and that $(\operatorname{val} f)_i K$ is an upper set). □

EXERCISE 1642. Write a simplified proof for the case if every $(\operatorname{form} f)_i$ is a join-semilattice.

LEMMA 1643. Every finitary prestaroid is completary.