CHAPTER 21

Multifuncoids and staroids

21.1. Product of two funcoids

21.1.1. Definition.

DEFINITION 1619. I will call a *quasi-invertible category* a partially ordered dagger category such that it holds

$$g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{\dagger} \tag{32}$$

for every morphisms $f \in \operatorname{Hom}(A, B)$, $g \in \operatorname{Hom}(B, C)$, $h \in \operatorname{Hom}(A, C)$, where A, B, C are objects of this category.

Inverting this formula, we get $f^{\dagger} \circ g^{\dagger} \not\asymp h^{\dagger} \Leftrightarrow g^{\dagger} \not\asymp f \circ h^{\dagger}$. After replacement of variables, this gives: $f^{\dagger} \circ g \not\asymp h \Leftrightarrow g \not\asymp f \circ h$.

EXERCISE 1620. Prove that every ordered groupoid is quasi-invertible category if we define the dagger as the inverse morphism.

As it follows from above, the categories **Rel** of binary relations (proposition 280), FCD of funcoids (theorem 879) and RLD of reloids (theorem 1006) are quasi-invertible (taking $f^{\dagger} = f^{-1}$). Moreover the category of pointfree funcoids between lattices of filters on boolean lattices is quasi-invertible (theorem 1551).

DEFINITION 1621. The *cross-composition product* of morphisms f and g of a quasi-invertible category is the pointfree funcoid $\operatorname{Hom}(\operatorname{Src} f, \operatorname{Src} g) \to \operatorname{Hom}(\operatorname{Dst} f, \operatorname{Dst} g)$ defined by the formulas (for every $a \in \operatorname{Hom}(\operatorname{Src} f, \operatorname{Src} g)$ and $b \in \operatorname{Hom}(\operatorname{Dst} f, \operatorname{Dst} g)$):

$$\left\langle f \times^{(C)} g \right\rangle a = g \circ a \circ f^{\dagger} \quad \text{and} \quad \left\langle (f \times^{(C)} g)^{-1} \right\rangle b = g^{\dagger} \circ b \circ f.$$

We need to prove that it is really a pointfree funcoid that is that

$$b \not\asymp \left\langle f \times^{(C)} g \right\rangle a \Leftrightarrow a \not\asymp \left\langle (f \times^{(C)} g)^{-1} \right\rangle b.$$

This formula means $b \not\asymp g \circ a \circ f^{\dagger} \Leftrightarrow a \not\asymp g^{\dagger} \circ b \circ f$ and can be easily proved applying formula (32) twice.

PROPOSITION 1622. $a \left[f \times^{(C)} g \right] b \Leftrightarrow a \circ f^{\dagger} \not\asymp g^{\dagger} \circ b$.

PROOF. From the definition. □

PROPOSITION 1623. $a \left[f \times^{(C)} g \right] b \Leftrightarrow f \left[a \times^{(C)} b \right] g$.

PROOF. $f \left[a \times^{(C)} b \right] g \Leftrightarrow f \circ a^{\dagger} \not\asymp b^{\dagger} \circ g \Leftrightarrow a \circ f^{\dagger} \not\asymp g^{\dagger} \circ b \Leftrightarrow a \left[f \times^{(C)} g \right] b$. □

THEOREM 1624. $(f \times^{(C)} g)^{-1} = f^{\dagger} \times^{(C)} g^{\dagger}$.

PROOF. For every morphisms $a \in \operatorname{Hom}(\operatorname{Src} f, \operatorname{Src} g)$ and $b \in \operatorname{Hom}(\operatorname{Dst} f, \operatorname{Dst} g)$ we have:

$\langle (f \times^{(C)} g)^{-1} \rangle b = g^{\dagger} \circ b \circ f = \langle f^{\dagger} \times^{(C)} g^{\dagger} \rangle b$.

$\langle ((f \times^{(C)} g)^{-1})^{-1} \rangle a = \langle f \times^{(C)} g \rangle a = g \circ a \circ f^{\dagger} = \langle (f^{\dagger} \times^{(C)} g^{\dagger})^{-1} \rangle a$. □