Claim: $\left(X \mapsto \bigsqcup_{x\in\mathscr{T}X\setminus\{\bot\}} \neg f_0 x, Y \mapsto \bigsqcup_{y\in\mathscr{T}Y\setminus\{\bot\}} f_1\neg y\right) = \left(X \mapsto \bigsqcup_{x\in X} \neg f_0\{x\}, Y \mapsto \bigsqcup_{y\in Y} f_1\neg\{y\}\right)$.

Proof: It is enough to prove $\bigsqcup_{x\in\mathscr{T}X\setminus\{\bot\}} \neg f_0 x = \bigsqcup_{x\in X} \neg f_0\{x\}$ for a Galois connection f (the rest follows from symmetry).

$\bigsqcup_{x\in\mathscr{T}X\setminus\{\bot\}} \neg f_0 x \sqsupseteq \bigsqcup_{x\in X} \neg f_0\{x\}$ because $\{x\} \in \mathscr{T}X \setminus \{\bot\}$. If $x \in \mathscr{T}X \setminus \{\bot\}$ then there exists $x' \in \{x\}$ and thus $\neg f_0\{x'\} \sqsupseteq \neg f_0 x$. Thus $\neg f_0 x \sqsubseteq \bigsqcup_{x\in X} \neg f_0\{x\}$ and so $\bigsqcup_{x\in\mathscr{T}X\setminus\{\bot\}} \neg f_0 x \sqsubseteq \bigsqcup_{x\in X} \neg f_0\{x\}$. ■

Claim: Ψ_5 is self-inverse.

Proof: Obvious. ■

Claim: $\Psi_4 = \Psi_5 \circ \Psi_3$.

Proof: Easily follows from symmetry. ■

Claim: $\Psi_4^{-1} = \Psi_3^{-1} \circ \Psi_5^{-1}$.

Proof: Easily follows from symmetry. ■

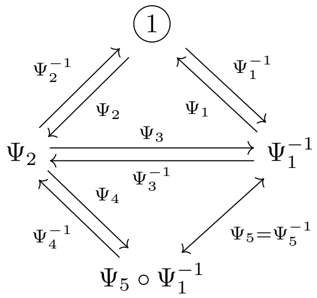
Claim: Ψ_4 and Ψ_4^{-1} are mutually inverse.

Proof: From two above claims and the fact that Ψ_3^{-1} is the inverse of Ψ_3 and Ψ_5^{-1} is the inverse of Ψ_5 proved above. ■

Note that now we have proved that Ψ_i and Ψ_i^{-1} are mutually inverse for all $i = 1, 2, 3, 4, 5$.

Claim: For every path of the diagram on figure 2 started with the circled node, the corresponding morphism is with which the node is labeled.

Figure 2.



Proof: Take into account that $\Psi_3^{-1} = \Psi_2 \circ \Psi_1$, $\Psi_4 = \Psi_5 \circ \Psi_3$ and thus also $\Psi_4 \circ \Psi_2 = \Psi_5 \circ \Psi_1^{-1}$. Now prove it by induction on path length. ■

Claim: Every cycle in the diagram at figure 1 is identity.

Proof: For cycles starting at the top node it follows from the previous claim. For arbitrary cycles it follows from theorem 192. ■

Claim: The diagram at figure 1 is commutative.

Proof: From the previous claim. ■

□

Proposition 1613. We equate the set of binary relations between A and B with $\mathbf{Rld}(A, B)$. Ψ_2 and Ψ_2^{-1} from the diagram at figure 1 preserve composition and identities (that are functors between categories $\mathbf{Rel}$ and $(A, B) \mapsto \mathsf{pFCD}(\mathscr{T}A, \mathscr{T}B)$) and also reversal ($f \mapsto f^{-1}$).

Proof. Let $\langle f\rangle = \langle p\rangle^*$ and $\langle g\rangle = \langle q\rangle^*$. Then $\langle g \circ f\rangle = \langle g\rangle \circ \langle f\rangle = \langle q\rangle^* \circ \langle p\rangle^* = \langle q \circ p\rangle^*$. Likewise $\left\langle (g \circ f)^{-1}\right\rangle = \left\langle (q \circ p)^{-1}\right\rangle^*$. So Φ_2 preserves composition.

Let $p = 1^A_{\mathbf{Rel}}$ for some set A. Then $\langle f\rangle = \langle p\rangle^* = \left\langle 1^A_{\mathbf{Rel}}\right\rangle^* = \mathrm{id}_{\mathscr{P}A}$ and likewise $\left\langle f^{-1}\right\rangle = \mathrm{id}_{\mathscr{P}A}$, that is f is an identity pointfree funcoid. So Φ_2 preserves identities.