CHAPTER 20

Alternative representations of binary relations

THEOREM 1612. Let A and B be fixed sets. The diagram at the figure 1 is a commutative diagram (in category **Set**), every arrow in this diagram is an isomorphism. Every cycle in this diagram is an identity. All "parallel" arrows are mutually inverse.

For a Galois connection f I denote f_0 the lower adjoint and f_1 the upper adjoint. For simplicity, in the diagram I equate $\mathscr{P}A$ and $\mathscr{T}A$.

PROOF. First, note that despite we use the notation Ψ_i^{-1}, it is not yet proved that Ψ_i^{-1} is the inverse of Ψ_i. We will prove it below.

Now prove a list of claims. First concentrate on the upper "triangle" of the diagram (the lower one will be considered later).

Claim: $\left\{ \frac{(x,y)}{y \in f_0\{x\}} \right\} = \left\{ \frac{(x,y)}{x \in f_1\{y\}} \right\}$ when f is an antitone Galois connection between $\mathscr{P}A$ and $\mathscr{P}B$.

Proof: $y \in f_0\{x\} \Leftrightarrow \{y\} \sqsubseteq f_0\{x\} \Leftrightarrow \{x\} \sqsubseteq f_1\{y\} \Leftrightarrow x \in f_1\{y\}$. ■

Claim: $\left(X \mapsto \bigsqcap_{x \in \mathscr{T}X \setminus \{\bot\}} \langle f \rangle x, Y \mapsto \bigsqcap_{y \in \mathscr{T}Y \setminus \{\bot\}} \langle f^{-1} \rangle y\right) = \left(X \mapsto \bigsqcap_{x \in X} \langle f \rangle \{x\}, Y \mapsto \bigsqcap_{y \in Y} \langle f^{-1} \rangle \{y\}\right)$ when f is a pointfree funcoid between $\mathscr{P}A$ and $\mathscr{P}B$.

Proof: It is enough to prove $\bigsqcap_{x \in \mathscr{T}X \setminus \{\bot\}} \langle f \rangle x = \bigsqcap_{x \in X} \langle f \rangle \{x\}$ (the rest follows from symmetry). $\bigsqcap_{x \in \mathscr{T}X \setminus \{\bot\}} \langle f \rangle x \sqsubseteq \bigsqcap_{x \in X} \langle f \rangle \{x\}$ because $\mathscr{T}X \setminus \{\bot\} \supseteq \left\{ \frac{\{x\}}{x \in X} \right\}$. $\bigsqcap_{x \in \mathscr{T}X \setminus \{\bot\}} \langle f \rangle x \sqsupseteq \bigsqcap_{x \in X} \langle f \rangle \{x\}$ because if $x \in \mathscr{T}X \setminus \{\bot\}$ then we can take $x' \in x$ that is $\{x'\} \subseteq x$ and thus $\langle f \rangle x \sqsupseteq \langle f \rangle \{x'\}$, so $\bigsqcap_{x \in \mathscr{T}X \setminus \{\bot\}} \langle f \rangle x \sqsupseteq \bigsqcap_{x \in \mathscr{T}X \setminus \{\bot\}} \langle f \rangle \{x'\} \sqsupseteq \bigsqcap_{x \in X} \langle f \rangle \{x\}$. ■

Claim: $\left(\mathscr{P}A, \mathscr{P}B, X \mapsto \bigsqcup_{x \in \mathscr{T}X \setminus \{\bot\}} f_0 x, Y \mapsto \bigsqcup_{y \in \mathscr{T}Y \setminus \{\bot\}} f_1 y\right) =$
$\left(\mathscr{P}A, \mathscr{P}B, X \mapsto \bigsqcup_{x \in X} f_0\{x\}, Y \mapsto \bigsqcup_{y \in Y} f_1\{y\}\right)$ when f is an antitone Galois connection between $\mathscr{P}A$ and $\mathscr{P}B$.

Proof: It is enough to prove $\bigsqcup_{x \in \mathscr{T}X \setminus \{\bot\}} f_0 x = \bigsqcup_{x \in X} f_0\{x\}$ (the rest follows from symmetry). We have $\bigsqcup_{x \in \mathscr{T}X \setminus \{\bot\}} f_0 x \sqsupseteq \bigsqcup_{x \in X} f_0\{x\}$ because $\{x\} \in \mathscr{T}X \setminus \{\bot\}$. Let $x \in \mathscr{T}X \setminus \{\bot\}$. Take $x' \in X$. We have $f_0 x \sqsubseteq f_0\{x'\}$ and thus $f_0 x \sqsubseteq \bigsqcup_{x \in X} f_0\{x\}$. So $\bigsqcup_{x \in \mathscr{T}X \setminus \{\bot\}} f_0 x \sqsubseteq \bigsqcup_{x \in X} f_0\{x\}$. ■

Claim: $\Psi_3^{-1} = \Psi_2 \circ \Psi_1$.

Proof: $\Psi_2 \Psi_1 f = \left(\mathscr{P}A, \mathscr{P}B, X \mapsto \left\{ \frac{y}{\exists x \in X : (x,y) \in \Psi_1 f} \right\}, Y \mapsto \left\{ \frac{x}{\exists y \in Y : (x,y) \in \Psi_1 f} \right\}\right) =$
$\left(\mathscr{P}A, \mathscr{P}B, X \mapsto \left\{ \frac{y}{\exists x \in X : y \in f_0\{x\}} \right\}, Y \mapsto \left\{ \frac{x}{\exists y \in Y : x \in f_1\{y\}} \right\}\right) =$
$\left(\mathscr{P}A, \mathscr{P}B, X \mapsto \bigsqcup_{x \in X} f_0\{x\}, Y \mapsto \bigsqcup_{y \in Y} f_1\{y\}\right) = \Psi_3^{-1} f$. ■

Claim: $\Psi_3 = \Psi_1^{-1} \circ \Psi_2^{-1}$.

Proof: $\Psi_1^{-1} \Psi_2^{-1} f = \left(X \mapsto \left\{ \frac{y \in B}{\forall x \in X : \{x\}[f]\{y\}} \right\}, Y \mapsto \left\{ \frac{x \in A}{\forall y \in Y : \{x\}[f]\{y\}} \right\}\right) =$
$\left(X \mapsto \left\{ \frac{y \in B}{\forall x \in X : y \in \langle f \rangle \{x\}} \right\}, Y \mapsto \left\{ \frac{x \in A}{\forall y \in Y : x \in \langle f^{-1} \rangle \{y\}} \right\}\right) =$
$\left(X \mapsto \bigsqcap_{x \in X} \langle f \rangle \{x\}, Y \mapsto \bigsqcap_{y \in Y} \langle f^{-1} \rangle \{y\}\right) = \Psi_3 f$. ■