It is tempting to conclude that $\langle f\rangle$ is a lower adjoint to $\langle f^{-1}\rangle$. But that's false: We should disprove that $\langle f\rangle X \sqsubseteq Y \Leftrightarrow X \sqsubseteq \langle f^{-1}\rangle Y$.

For a counter-example, take $f = \{0\} \times \mathbb{N}$. Then our condition takes form $Y = \mathbb{N} \Leftrightarrow X \sqsubseteq \{0\}$ for $X \ni 0$, $Y \ni 0$ what obviously does not hold.

## 19.17. Binary relations are pointfree funcoids

Below for simplicity we will equate $\mathscr{T} A$ with $\mathscr{P} A$.

THEOREM 1606. Pointfree funcoids $f$ between powerset posets $\mathscr{T} A$ and $\mathscr{T} B$ bijectively (moreover this bijection is an order-isomorphism) correspond to morphisms $p \in \mathbf{Rel}(A, B)$ by the formulas:

$$\langle f\rangle = \langle p\rangle^*, \quad \langle f^{-1}\rangle = \langle p^{-1}\rangle^*; \tag{30}$$

$$(x, y) \in \operatorname{GR} p \Leftrightarrow y \in \langle f\rangle\{x\} \Leftrightarrow x \in \langle f^{-1}\rangle\{y\}. \tag{31}$$

PROOF. Suppose $p \in \mathbf{Rel}(A, B)$ and prove that there is a pointfree funcoid $f$ conforming to (30). Really, for every $X \in \mathscr{T} A$, $Y \in \mathscr{T} B$

$$Y \not\asymp \langle f\rangle X \Leftrightarrow Y \not\asymp \langle p\rangle^* X \Leftrightarrow Y \not\asymp \langle p\rangle X \Leftrightarrow$$
$$X \not\asymp \langle p^{-1}\rangle Y \Leftrightarrow X \not\asymp \langle p^{-1}\rangle^* Y \Leftrightarrow X \not\asymp \langle f^{-1}\rangle Y.$$

Now suppose $f \in \mathsf{pFCD}(\mathscr{T} A, \mathscr{T} B)$ and prove that the relation defined by the formula (31) exists. To prove it, it's enough to show that $y \in \langle f\rangle\{x\} \Leftrightarrow x \in \langle f^{-1}\rangle\{y\}$. Really,

$$y \in \langle f\rangle\{x\} \Leftrightarrow \{y\} \not\asymp \langle f\rangle\{x\} \Leftrightarrow \{x\} \not\asymp \langle f^{-1}\rangle\{y\} \Leftrightarrow x \in \langle f^{-1}\rangle\{y\}.$$

It remains to prove that functions defined by (30) and (31) are mutually inverse. (That these functions are monotone is obvious.)

Let $p_0 \in \mathbf{Rel}(A, B)$ and $f \in \mathsf{pFCD}(\mathscr{T} A, \mathscr{T} B)$ corresponds to $p_0$ by the formula (30); let $p_1 \in \mathbf{Rel}(A, B)$ corresponds to $f$ by the formula (31). Then $p_0 = p_1$ because

$$(x, y) \in \operatorname{GR} p_0 \Leftrightarrow y \in \langle p_0\rangle^*\{x\} \Leftrightarrow y \in \langle f\rangle\{x\} \Leftrightarrow (x, y) \in \operatorname{GR} p_1.$$

Let now $f_0 \in \mathsf{pFCD}(\mathscr{T} A, \mathscr{T} B)$ and $p \in \mathbf{Rel}(A, B)$ corresponds to $f_0$ by the formula (31); let $f_1 \in \mathsf{pFCD}(\mathscr{T} A, \mathscr{T} B)$ corresponds to $p$ by the formula (30). Then $(x, y) \in \operatorname{GR} p \Leftrightarrow y \in \langle f_0\rangle\{x\}$ and $\langle f_1\rangle = \langle p\rangle^*$; thus

$$y \in \langle f_1\rangle\{x\} \Leftrightarrow y \in \langle p\rangle^*\{x\} \Leftrightarrow (x, y) \in \operatorname{GR} p \Leftrightarrow y \in \langle f_0\rangle\{x\}.$$

So $\langle f_0\rangle = \langle f_1\rangle$. Similarly $\langle f_0^{-1}\rangle = \langle f_1^{-1}\rangle$. □

PROPOSITION 1607. The bijection defined by the theorem 1606 preserves composition and identities, that is is a functor between categories $\mathbf{Rel}$ and $(A, B) \mapsto \mathsf{pFCD}(\mathscr{T} A, \mathscr{T} B)$.

PROOF. Let $\langle f\rangle = \langle p\rangle^*$ and $\langle g\rangle = \langle q\rangle^*$. Then $\langle g \circ f\rangle = \langle g\rangle \circ \langle f\rangle = \langle q\rangle^* \circ \langle p\rangle^* = \langle q \circ p\rangle^*$. Likewise $\langle (g \circ f)^{-1}\rangle = \langle (q \circ p)^{-1}\rangle^*$. So it preserves composition.

Let $p = 1^A_{\mathbf{Rel}}$ for some set $A$. Then $\langle f\rangle = \langle p\rangle^* = \langle 1^A_{\mathbf{Rel}}\rangle^* = \mathrm{id}_{\mathscr{P} A}$ and likewise $\langle f^{-1}\rangle = \mathrm{id}_{\mathscr{P} A}$, that is $f$ is an identity pointfree funcoid. So it preserves identities. □

PROPOSITION 1608. The bijection defined by theorem 1606 preserves reversal.

PROOF. $\langle f^{-1}\rangle = \langle p^{-1}\rangle^*$. □

PROPOSITION 1609. The bijection defined by theorem 1606 preserves monovaluedness and injectivity.