$\Leftarrow$.

$$\begin{aligned}&\langle f^{-1}\rangle(\mathcal{I}\sqcap\mathcal{J})=\\&\bigsqcap\langle\langle f^{-1}\rangle\rangle^{*}\operatorname{up}^{\mathfrak{Z}_1}(\mathcal{I}\sqcap\mathcal{J})=\\&\bigsqcap\langle\langle f^{-1}\rangle\rangle^{*}\left\{\frac{I\sqcap^{\mathfrak{Z}_1}J}{I\in\operatorname{up}\mathcal{I},J\in\operatorname{up}\mathcal{J}}\right\}=\\&\bigsqcap\left\{\frac{\langle f^{-1}\rangle(I\sqcap^{\mathfrak{Z}_1}J)}{I\in\operatorname{up}\mathcal{I},J\in\operatorname{up}\mathcal{J}}\right\}=\\&\bigsqcap\left\{\frac{\langle f^{-1}\rangle I\sqcap\langle f^{-1}\rangle J}{I\in\operatorname{up}\mathcal{I},J\in\operatorname{up}\mathcal{J}}\right\}=\\&\bigsqcap\left\{\frac{\langle f^{-1}\rangle I}{I\in\operatorname{up}\mathcal{I}}\right\}\sqcap\bigsqcap\left\{\frac{\langle f^{-1}\rangle J}{J\in\operatorname{up}\mathcal{J}}\right\}=\\&\langle f^{-1}\rangle\mathcal{I}\sqcap^{\mathfrak{A}}\langle f^{-1}\rangle\mathcal{J}\end{aligned}$$

(used theorem 1509, corollary 518, theorem 1498).

□

PROPOSITION 1593. Let $\mathfrak{A}$ be an atomistic meet-semilattice with least element, $\mathfrak{B}$ be an atomistic bounded meet-semilattice. Then if f, g are pointfree funcoids from $\mathfrak{A}$ to $\mathfrak{B}$, $f\sqsubseteq g$ and g is monovalued then $g|_{\operatorname{dom} f}=f$.

PROOF. Obviously $g|_{\operatorname{dom} f}\sqsupseteq f$. Suppose for contrary that $g|_{\operatorname{dom} f}\sqsubset f$. Then there exists an atom $a\in\operatorname{atoms}\operatorname{dom} f$ such that $\langle g|_{\operatorname{dom} f}\rangle a\neq\langle f\rangle a$ that is $\langle g\rangle a\sqsubset\langle f\rangle a$ what is impossible. □

19.14. Elements closed regarding a pointfree funcoid

Let $\mathfrak{A}$ be a poset. Let $f\in\mathsf{pFCD}(\mathfrak{A},\mathfrak{A})$.

DEFINITION 1594. Let's call *closed* regarding a pointfree funcoid f such element $a\in\mathfrak{A}$ that $\langle f\rangle a\sqsubseteq a$.

PROPOSITION 1595. If i and j are closed (regarding a pointfree funcoid $f\in\mathsf{pFCD}(\mathfrak{A},\mathfrak{A})$), S is a set of closed elements (regarding f), then

1°. $i\sqcup j$ is a closed element, if $\mathfrak{A}$ is a separable starrish join-semilattice;
2°. $\bigsqcap S$ is a closed element if $\mathfrak{A}$ is a strongly separable complete lattice.

PROOF. $\langle f\rangle(i\sqcup j)=\langle f\rangle i\sqcup\langle f\rangle j\sqsubseteq i\sqcup j$ (theorem 1498), $\langle f\rangle\bigsqcap S\sqsubseteq\bigsqcap\langle\langle f\rangle\rangle^{*}S\sqsubseteq\bigsqcap S$ (used strong separability of $\mathfrak{A}$ twice). Consequently the elements $i\sqcup j$ and $\bigsqcap S$ are closed. □

PROPOSITION 1596. If S is a set of elements closed regarding a complete pointfree funcoid f with strongly separable destination which is a complete lattice, then the element $\bigsqcup S$ is also closed regarding our funcoid.

PROOF. $\langle f\rangle\bigsqcup S=\bigsqcup\langle\langle f\rangle\rangle^{*}S\sqsubseteq\bigsqcup S$. □

19.15. Connectedness regarding a pointfree funcoid

Let $\mathfrak{A}$ be a poset with least element. Let $\mu\in\mathsf{pFCD}(\mathfrak{A},\mathfrak{A})$.

DEFINITION 1597. An element $a\in\mathfrak{A}$ is called *connected* regarding a pointfree funcoid μ over $\mathfrak{A}$ when

$$\forall x,y\in\mathfrak{A}\setminus\{\bot^{\mathfrak{A}}\}:(x\sqcup y=a\Rightarrow x\ [\mu]\ y).$$