Obvious 1586. Co-completion is always co-complete.

Proposition 1587. For above defined always $\operatorname{CoCompl} f \sqsubseteq f$.

Proof. By proposition 539. □

19.13. Monovalued and injective pointfree funcoids

Definition 1588. Let $\mathfrak{A}$ and $\mathfrak{B}$ be posets. Let $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$.
The pointfree funcoid f is:

- *monovalued* when $f \circ f^{-1} \sqsubseteq 1^{\mathsf{pFCD}}_{\mathfrak{B}}$.
- *injective* when $f^{-1} \circ f \sqsubseteq 1^{\mathsf{pFCD}}_{\mathfrak{A}}$.

Monovaluedness is dual of injectivity.

Proposition 1589. Let $\mathfrak{A}$ and $\mathfrak{B}$ be posets. Let $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$.
The pointfree funcoid f is:

- monovalued iff $f \circ f^{-1} \sqsubseteq \mathrm{id}^{\mathsf{pFCD}(\mathfrak{B})}_{\operatorname{im} f}$, if $\mathfrak{A}$ has greatest element and $\mathfrak{B}$ is a strongly separable meet-semilattice;
- injective iff $f^{-1} \circ f \sqsubseteq \mathrm{id}^{\mathsf{pFCD}(\mathfrak{A})}_{\operatorname{dom} f}$, if $\mathfrak{B}$ has greatest element and $\mathfrak{A}$ is a strongly separable meet-semilattice.

Proof. It's enough to prove $f \circ f^{-1} \sqsubseteq 1^{\mathsf{pFCD}}_{\mathfrak{B}} \Leftrightarrow f \circ f^{-1} \sqsubseteq \mathrm{id}^{\mathsf{pFCD}(\mathfrak{B})}_{\operatorname{im} f}$. $\operatorname{im} f$ is defined because $\mathfrak{A}$ has greatest element. $\mathrm{id}^{\mathsf{pFCD}(\mathfrak{B})}_{\operatorname{im} f}$ is defined because $\mathfrak{B}$ is a meet-semilattice.

$\Leftarrow$. Obvious.

$\Rightarrow$. Let $f \circ f^{-1} \sqsubseteq 1^{\mathsf{pFCD}}_{\mathfrak{B}}$. Then $\langle f \circ f^{-1} \rangle x \sqsubseteq x$; $\langle f \circ f^{-1} \rangle x \sqsubseteq \operatorname{im} f$ (proposition 1497). Thus $\langle f \circ f^{-1} \rangle x \sqsubseteq x \sqcap \operatorname{im} f = \left\langle \mathrm{id}^{\mathsf{pFCD}(\mathfrak{B})}_{\operatorname{im} f} \right\rangle x$.

$\langle (f \circ f^{-1})^{-1} \rangle x \sqsubseteq x$ and $\langle (f \circ f^{-1})^{-1} \rangle x = \langle f \circ f^{-1} \rangle x \sqsubseteq \operatorname{im} f$. Thus $\langle (f \circ f^{-1})^{-1} \rangle x \sqsubseteq x \sqcap \operatorname{im} f = \left\langle \mathrm{id}^{\mathsf{pFCD}(\mathfrak{B})}_{\operatorname{im} f} \right\rangle x$.

Thus $f \circ f^{-1} \sqsubseteq \mathrm{id}^{\mathsf{pFCD}(\mathfrak{B})}_{\operatorname{im} f}$.

□

Theorem 1590. Let $\mathfrak{A}$ be an atomistic meet-semilattice with least element, $\mathfrak{B}$ be an atomistic bounded meet-semilattice. The following statements are equivalent for every $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$:

1°. f is monovalued.
2°. $\forall a \in \operatorname{atoms}^{\mathfrak{A}} : \langle f \rangle a \in \operatorname{atoms}^{\mathfrak{B}} \cup \{\bot^{\mathfrak{B}}\}$.
3°. $\forall i, j \in \mathfrak{B} : \langle f^{-1} \rangle (i \sqcap j) = \langle f^{-1} \rangle i \sqcap \langle f^{-1} \rangle j$.

Proof.

2°⇒3°. Let $a \in \operatorname{atoms}^{\mathfrak{A}}$, $\langle f \rangle a = b$. Then because $b \in \operatorname{atoms}^{\mathfrak{B}} \cup \{\bot^{\mathfrak{B}}\}$

$$(i \sqcap j) \sqcap b \neq \bot^{\mathfrak{B}} \Leftrightarrow i \sqcap b \neq \bot^{\mathfrak{B}} \wedge j \sqcap b \neq \bot^{\mathfrak{B}};$$
$$a \, [f] \, i \sqcap j \Leftrightarrow a \, [f] \, i \wedge a \, [f] \, j;$$
$$i \sqcap j \, \left[f^{-1}\right] \, a \Leftrightarrow i \, \left[f^{-1}\right] \, a \wedge j \, \left[f^{-1}\right] \, a;$$
$$a \sqcap^{\mathfrak{A}} \langle f^{-1} \rangle (i \sqcap j) \neq \bot^{\mathfrak{A}} \Leftrightarrow a \sqcap \langle f^{-1} \rangle i \neq \bot^{\mathfrak{A}} \wedge a \sqcap \langle f^{-1} \rangle j \neq \bot^{\mathfrak{A}};$$
$$a \sqcap^{\mathfrak{A}} \langle f^{-1} \rangle (i \sqcap j) \neq \bot^{\mathfrak{A}} \Leftrightarrow a \sqcap \langle f^{-1} \rangle i \sqcap \langle f^{-1} \rangle j \neq \bot^{\mathfrak{A}};$$
$$\langle f^{-1} \rangle (i \sqcap j) = \langle f^{-1} \rangle i \sqcap \langle f^{-1} \rangle j.$$

3°⇒1°. $\langle f^{-1} \rangle a \sqcap \langle f^{-1} \rangle b = \langle f^{-1} \rangle (a \sqcap b) = \langle f^{-1} \rangle \bot^{\mathfrak{B}} = \bot^{\mathfrak{A}}$ (by proposition 1496) for every two distinct $a, b \in \operatorname{atoms}^{\mathfrak{B}}$. This is equivalent to $\neg(\langle f^{-1} \rangle a \, [f]$