PROPOSITION 1581.

1°. Let $f \in \mathsf{ComplpFCD}(\mathfrak{A}, \mathfrak{B})$ and $g \in \mathsf{ComplpFCD}(\mathfrak{B}, \mathfrak{C})$ where $\mathfrak{A}$ and $\mathfrak{C}$ are posets with least elements and $\mathfrak{B}$ is a complete lattice. Then $g \circ f \in \mathsf{ComplpFCD}(\mathfrak{A}, \mathfrak{C})$.

2°. Let $f \in \mathsf{CoComplpFCD}(\mathfrak{A}, \mathfrak{B})$ and $g \in \mathsf{CoComplpFCD}(\mathfrak{B}, \mathfrak{C})$ where $(\mathfrak{A}, \mathfrak{Z}_0)$, $(\mathfrak{B}, \mathfrak{Z}_1)$, $(\mathfrak{C}, \mathfrak{Z}_2)$ are filtrators. Then $g \circ f \in \mathsf{CoComplpFCD}(\mathfrak{A}, \mathfrak{C})$.

PROOF.

1°. Let $\bigsqcup S$ and $\bigsqcup \langle\langle g \circ f\rangle\rangle^* S$ be defined. Then

$$\langle g \circ f\rangle \bigsqcup S = \langle g\rangle\langle f\rangle \bigsqcup S = \langle g\rangle \bigsqcup \langle\langle f\rangle\rangle^* S = \bigsqcup \langle\langle g\rangle\rangle^* \langle\langle f\rangle\rangle^* S = \bigsqcup \langle\langle g \circ f\rangle\rangle^* S.$$

2°. $\langle g \circ f\rangle \mathfrak{Z}_0 = \langle g\rangle\langle f\rangle \mathfrak{Z}_0 \in \mathfrak{Z}_2$ because $\langle f\rangle \mathfrak{Z}_0 \in \mathfrak{Z}_1$.

□

PROPOSITION 1582. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. Then $\mathsf{CoComplpFCD}(\mathfrak{A}, \mathfrak{B})$ (with induced order) is a complete lattice.

PROOF. Follows from the theorem 1580. □

THEOREM 1583. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators where $\mathfrak{Z}_0$ and $\mathfrak{Z}_1$ are boolean lattices. Let R be a set of pointfree funcoids from $\mathfrak{A}$ to $\mathfrak{B}$.

$g \circ (\bigsqcup R) = \bigsqcup_{g \in R} (g \circ f) = \bigsqcup \langle g \circ\rangle^* R$ if g is a complete pointfree funcoid from $\mathfrak{B}$.

PROOF. For every $X \in \mathfrak{A}$

$$\begin{aligned} \left\langle g \circ \left(\bigsqcup R\right)\right\rangle X &= \\ \langle g\rangle \left\langle \bigsqcup R\right\rangle X &= \\ \langle g\rangle \bigsqcup_{f \in R} \langle f\rangle X &= \\ \bigsqcup_{f \in R} \langle g\rangle\langle f\rangle X &= \\ \bigsqcup_{f \in R} \langle g \circ f\rangle X &= \\ \left\langle \bigsqcup_{f \in R} (g \circ f)\right\rangle X &= \\ \left\langle \bigsqcup \langle g \circ\rangle^* R\right\rangle X. \end{aligned}$$

So $g \circ (\bigsqcup R) = \bigsqcup \langle g \circ\rangle^* R$. □

19.12. Completion and co-completion

DEFINITION 1584. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices and $\mathfrak{Z}_1$ is a complete atomistic lattice.

Co-completion of a pointfree funcoid $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ is pointfree funcoid $\operatorname{CoCompl} f$ defined by the formula (for every $X \in \mathfrak{Z}_0$)

$$\langle \operatorname{CoCompl} f\rangle X = \operatorname{Cor}\langle f\rangle X.$$

PROPOSITION 1585. Above defined co-completion always exists.

PROOF. Existence of $\operatorname{Cor}\langle f\rangle X$ follows from completeness of $\mathfrak{Z}_1$.

We may apply the theorem 1510 because

$$\operatorname{Cor}\langle f\rangle (X \sqcup^{\mathfrak{Z}_0} Y) = \operatorname{Cor}(\langle f\rangle X \sqcup^{\mathfrak{B}} \langle f\rangle Y) = \operatorname{Cor}\langle f\rangle X \sqcup^{\mathfrak{Z}_1} \operatorname{Cor}\langle f\rangle Y$$

by theorem 600. □