Proof.

$$\begin{aligned}(x \times^{\mathsf{FCD}} z) \sqcap (g \circ f) \neq \bot^{\mathsf{pFCD}(\mathfrak{A},\mathfrak{C})} &\Leftrightarrow \\ x \mathrel{[g \circ f]} z &\Leftrightarrow \\ \exists y \in \operatorname{atoms}^{\mathfrak{B}} : (x \mathrel{[f]} y \wedge y \mathrel{[g]} z) &\Leftrightarrow \\ \exists y \in \operatorname{atoms}^{\mathfrak{B}} : ((x \times^{\mathsf{FCD}} y) \sqcap f \neq \bot^{\mathsf{pFCD}(\mathfrak{A},\mathfrak{B})} \wedge (y \times^{\mathsf{FCD}} z) \sqcap g \neq \bot^{\mathsf{pFCD}(\mathfrak{B},\mathfrak{C})})\end{aligned}$$

(were used corollary 1558 and theorem 1549). □

Theorem 1575. Let f be a pointfree funcoid between atomic bounded separable meet-semilattices $\mathfrak{A}$ and $\mathfrak{B}$.

1°. $\mathcal{X} \mathrel{[f]} \mathcal{Y} \Leftrightarrow \exists F \in \operatorname{atoms} f : \mathcal{X} \mathrel{[F]} \mathcal{Y}$ for every $\mathcal{X} \in \mathfrak{A}$, $\mathcal{Y} \in \mathfrak{B}$;

2°. $\langle f \rangle \mathcal{X} = \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$ for every $\mathcal{X} \in \mathfrak{A}$ provided that $\mathfrak{B}$ is a complete lattice.

Proof.

1°.

$$\begin{aligned}\exists F \in \operatorname{atoms} f : \mathcal{X} \mathrel{[F]} \mathcal{Y} &\Leftrightarrow \\ \exists a \in \operatorname{atoms}^{\mathfrak{A}}, b \in \operatorname{atoms}^{\mathfrak{B}} : (a \times^{\mathsf{FCD}} b \not\asymp f \wedge \mathcal{X} \left[a \times^{\mathsf{FCD}} b\right] \mathcal{Y}) &\Leftrightarrow \\ \exists a \in \operatorname{atoms}^{\mathfrak{A}}, b \in \operatorname{atoms}^{\mathfrak{B}} : (a \times^{\mathsf{FCD}} b \not\asymp f \wedge a \times^{\mathsf{FCD}} b \not\asymp \mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y}) &\Leftrightarrow \\ \exists F \in \operatorname{atoms} f : (F \not\asymp f \wedge F \not\asymp \mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y}) &\Leftrightarrow \\ &\text{(by theorem 1570)} \\ f \not\asymp \mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y} &\Leftrightarrow \\ \mathcal{X} \mathrel{[f]} \mathcal{Y}.&\end{aligned}$$

2°. Let $\mathcal{Y} \in \mathfrak{B}$. Suppose $\mathcal{Y} \not\asymp \langle f \rangle \mathcal{X}$. Then $\mathcal{X} \mathrel{[f]} \mathcal{Y}$; $\exists F \in \operatorname{atoms} f : \mathcal{X} \mathrel{[F]} \mathcal{Y}$; $\exists F \in \operatorname{atoms} f : \mathcal{Y} \not\asymp \langle F \rangle \mathcal{X}$; and (taking into account that $\mathfrak{B}$ is strongly separable by theorem 222) $\mathcal{Y} \not\asymp \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$. So $\langle f \rangle \mathcal{X} \sqsubseteq \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$ by strong separability. The contrary $\langle f \rangle \mathcal{X} \sqsupseteq \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$ is obvious.

□

19.11. Complete pointfree funcoids

Definition 1576. Let $\mathfrak{A}$ and $\mathfrak{B}$ be posets. A pointfree funcoid $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ is *complete*, when for every $S \in \mathscr{P}\mathfrak{A}$ whenever both $\bigsqcup S$ and $\bigsqcup \langle \langle f \rangle \rangle^* S$ are defined we have

$$\langle f \rangle \bigsqcup S = \bigsqcup \langle \langle f \rangle \rangle^* S.$$

Definition 1577. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be filtrators. I will call a *co-complete pointfree funcoid* a pointfree funcoid $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ such that $\langle f \rangle X \in \mathfrak{Z}_1$ for every $X \in \mathfrak{Z}_0$.

Proposition 1578. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. Co-complete pointfree funcoids $\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ bijectively correspond to functions $\mathfrak{Z}_1^{\mathfrak{Z}_0}$ preserving finite joins, where the bijection is $f \mapsto \langle f \rangle|_{\mathfrak{Z}_0}$.

Proof. It follows from the theorem 1510. □

Theorem 1579. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ be a down-aligned, with join-closed, binarily meet-closed and separable core which is a complete boolean lattice.

Let $(\mathfrak{B}, \mathfrak{Z}_1)$ be a star-separable filtrator.

The following conditions are equivalent for every pointfree funcoid $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$: