19.10. Atomic pointfree funcoids

THEOREM 1569. Let $\mathfrak{A}$, $\mathfrak{B}$ be atomic bounded separable meet-semilattices. An $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ is an atom of the poset $\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ iff there exist $a \in \operatorname{atoms}^{\mathfrak{A}}$ and $b \in \operatorname{atoms}^{\mathfrak{B}}$ such that $f = a \times^{\mathsf{FCD}} b$.

PROOF.

$\Rightarrow$. Let f be an atom of the poset $\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$. Let's get elements $a \in \operatorname{atoms} \operatorname{dom} f$ and $b \in \operatorname{atoms}\langle f \rangle a$. Then for every $\mathcal{X} \in \mathfrak{A}$

$$\mathcal{X} \asymp a \Rightarrow \left\langle a \times^{\mathsf{FCD}} b \right\rangle \mathcal{X} = \bot^{\mathfrak{B}} \sqsubseteq \langle f \rangle \mathcal{X}, \quad \mathcal{X} \not\asymp a \Rightarrow \left\langle a \times^{\mathsf{FCD}} b \right\rangle \mathcal{X} = b \sqsubseteq \langle f \rangle \mathcal{X}.$$

So $\left\langle a \times^{\mathsf{FCD}} b \right\rangle \mathcal{X} \sqsubseteq \langle f \rangle \mathcal{X}$ and similarly $\left\langle b \times^{\mathsf{FCD}} a \right\rangle \mathcal{Y} \sqsubseteq \left\langle f^{-1} \right\rangle \mathcal{Y}$ for every $\mathcal{Y} \in \mathfrak{B}$ thus $a \times^{\mathsf{FCD}} b \sqsubseteq f$; because f is atomic we have $f = a \times^{\mathsf{FCD}} b$.

$\Leftarrow$. Let $a \in \operatorname{atoms}^{\mathfrak{A}}$, $b \in \operatorname{atoms}^{\mathfrak{B}}$, $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$. If $b \asymp^{\mathfrak{B}} \langle f \rangle a$ then $\neg(a\ [f]\ b)$, $f \sqcap (a \times^{\mathsf{FCD}} b) = \bot^{\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})}$ (by corollary 1558 because $\mathfrak{A}$ and $\mathfrak{B}$ are bounded meet-semilattices); if $b \sqsubseteq \langle f \rangle a$, then for every $\mathcal{X} \in \mathfrak{A}$

$$\mathcal{X} \asymp a \Rightarrow \left\langle a \times^{\mathsf{FCD}} b \right\rangle \mathcal{X} = \bot^{\mathfrak{B}} \sqsubseteq \langle f \rangle \mathcal{X}, \quad \mathcal{X} \not\asymp a \Rightarrow \left\langle a \times^{\mathsf{FCD}} b \right\rangle \mathcal{X} = b \sqsubseteq \langle f \rangle \mathcal{X}$$

that is $\left\langle a \times^{\mathsf{FCD}} b \right\rangle \mathcal{X} \sqsubseteq \langle f \rangle \mathcal{X}$ and likewise $\left\langle b \times^{\mathsf{FCD}} a \right\rangle \mathcal{Y} \sqsubseteq \left\langle f^{-1} \right\rangle \mathcal{Y}$ for every $\mathcal{Y} \in \mathfrak{B}$, so $f \sqsupseteq a \times^{\mathsf{FCD}} b$. Consequently $f \sqcap (a \times^{\mathsf{FCD}} b) = \bot^{\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})} \vee f \sqsupseteq a \times^{\mathsf{FCD}} b$; that is $a \times^{\mathsf{FCD}} b$ is an atomic pointfree funcoid.

□

THEOREM 1570. Let $\mathfrak{A}$, $\mathfrak{B}$ be atomic bounded separable meet-semilattices. Then $\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ is atomic.

PROOF. Let $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ and $f \neq \bot^{\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})}$. Then $\operatorname{dom} f \neq \bot^{\mathfrak{A}}$, thus exists $a \in \operatorname{atoms} \operatorname{dom} f$. So $\langle f \rangle a \neq \bot^{\mathfrak{B}}$ thus exists $b \in \operatorname{atoms}\langle f \rangle a$. Finally the atomic pointfree funcoid $a \times^{\mathsf{FCD}} b \sqsubseteq f$. □

PROPOSITION 1571. Let $\mathfrak{A}$, $\mathfrak{B}$ be starrish bounded separable lattices. $\operatorname{atoms}(f \sqcup g) = \operatorname{atoms} f \cup \operatorname{atoms} g$ for every $f, g \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$.

PROOF.

$$\begin{aligned}(a \times^{\mathsf{FCD}} b) \sqcap (f \sqcup g) \neq \bot^{\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})} &\Leftrightarrow \text{(corollary 1558)} \Leftrightarrow \\ a\ [f \sqcup g]\ b &\Leftrightarrow \text{(theorem 1526)} \Leftrightarrow \\ a\ [f]\ b \vee a\ [g]\ b &\Leftrightarrow \text{(corollary 1558)} \Leftrightarrow \\ (a \times^{\mathsf{FCD}} b) \sqcap f \neq \bot^{\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})} &\vee (a \times^{\mathsf{FCD}} b) \sqcap g \neq \bot^{\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})}\end{aligned}$$

for every $a \in \operatorname{atoms}^{\mathfrak{A}}$ and $b \in \operatorname{atoms}^{\mathfrak{B}}$. □

THEOREM 1572. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. Then $\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ is a co-frame.

PROOF. Theorems 1510 and 530. □

COROLLARY 1573. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. Then $\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ is a co-brouwerian lattice.

PROPOSITION 1574. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be atomic bounded separable meet-semilattices, and $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$, $g \in \mathsf{pFCD}(\mathfrak{B}, \mathfrak{C})$. Then

$$\operatorname{atoms}(g \circ f) = \left\{ \frac{x \times^{\mathsf{FCD}} z}{x \in \operatorname{atoms}^{\mathfrak{A}}, z \in \operatorname{atoms}^{\mathfrak{C}}, \exists y \in \operatorname{atoms}^{\mathfrak{B}} : (x \times^{\mathsf{FCD}} y \in \operatorname{atoms} f \wedge y \times^{\mathsf{FCD}} z \in \operatorname{atoms} g)} \right\}.$$