PROPOSITION 1566. Let $\mathfrak{A}$ be a meet-semilattice with least element and $\mathfrak{B}$ be a poset with least element. If a is an atom of $\mathfrak{A}$, $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ then $f|_a = a \times^{\mathsf{FCD}} \langle f \rangle a$.

PROOF. Let $\mathcal{X} \in \mathfrak{A}$.

$$\mathcal{X} \sqcap a \neq \bot^{\mathfrak{A}} \Rightarrow \langle f|_a \rangle \mathcal{X} = \langle f \rangle a, \quad \mathcal{X} \sqcap a = \bot^{\mathfrak{A}} \Rightarrow \langle f|_a \rangle \mathcal{X} = \bot^{\mathfrak{B}}.$$

□

PROPOSITION 1567. $f \circ (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = \mathcal{A} \times^{\mathsf{FCD}} \langle f \rangle \mathcal{B}$ for elements $\mathcal{A} \in \mathfrak{A}$ and $B \in \mathfrak{B}$ of some posets $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ with least elements and $f \in \mathsf{pFCD}(\mathfrak{B}, \mathfrak{C})$.

PROOF. Let $\mathcal{X} \in \mathfrak{A}$, $\mathcal{Y} \in \mathfrak{B}$.

$$\langle f \circ (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \rangle \mathcal{X} = \left(\begin{cases} \langle f \rangle \mathcal{B} & \text{if } \mathcal{X} \not\asymp \mathcal{A} \\ \bot & \text{if } \mathcal{X} \asymp \mathcal{A} \end{cases} \right) = \langle \mathcal{A} \times^{\mathsf{FCD}} \langle f \rangle \mathcal{B} \rangle \mathcal{X};$$

$$\begin{gathered} \langle (f \circ (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}))^{-1} \rangle \mathcal{Y} = \\ \langle (\mathcal{B} \times^{\mathsf{FCD}} \mathcal{A}) \circ f^{-1} \rangle \mathcal{Y} = \\ \left(\begin{cases} \mathcal{A} & \text{if } \langle f^{-1} \rangle \mathcal{Y} \not\asymp \mathcal{B} \\ \bot & \text{if } \langle f^{-1} \rangle \mathcal{Y} \asymp \mathcal{B} \end{cases} \right) = \\ \left(\begin{cases} \mathcal{A} & \text{if } \mathcal{Y} \not\asymp \langle f \rangle \mathcal{B} \\ \bot & \text{if } \mathcal{Y} \asymp \langle f \rangle \mathcal{B} \end{cases} \right) = \\ \langle \langle f \rangle \mathcal{B} \times^{\mathsf{FCD}} \mathcal{A} \rangle \mathcal{Y} = \\ \langle (\mathcal{A} \times^{\mathsf{FCD}} \langle f \rangle \mathcal{B})^{-1} \rangle \mathcal{Y}. \end{gathered}$$

□

19.9. Category of pointfree funcoids

I will define the category pFCD of pointfree funcoids:

- The class of objects are small posets.
- The set of morphisms from $\mathfrak{A}$ to $\mathfrak{B}$ is $\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$.
- The composition is the composition of pointfree funcoids.
- Identity morphism for an object $\mathfrak{A}$ is $(\mathfrak{A}, \mathfrak{A}, \mathrm{id}_{\mathfrak{A}}, \mathrm{id}_{\mathfrak{A}})$.

To prove that it is really a category is trivial.

The *category of pointfree funcoid quintuples* is defined as follows:

- Objects are pairs $(\mathfrak{A}, \mathcal{A})$ where $\mathfrak{A}$ is a small meet-semilattice and $\mathcal{A} \in \mathfrak{A}$.
- The morphisms from an object $(\mathfrak{A}, \mathcal{A})$ to an object $(\mathfrak{B}, \mathcal{B})$ are tuples $(\mathfrak{A}, \mathfrak{B}, \mathcal{A}, \mathcal{B}, f)$ where $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ and
$$\forall x \in \mathfrak{A} : \langle f \rangle x \sqsubseteq \mathcal{A}, \quad \forall y \in \mathfrak{B} : \langle f^{-1} \rangle y \sqsubseteq \mathcal{B}. \tag{28}$$
- The composition is defined by the formula
$$(\mathfrak{B}, \mathfrak{C}, \mathcal{B}, \mathcal{C}, g) \circ (\mathfrak{A}, \mathfrak{B}, \mathcal{A}, \mathcal{B}, f) = (\mathfrak{A}, \mathfrak{C}, \mathcal{A}, \mathcal{C}, g \circ f).$$
- Identity morphism for an object $(\mathfrak{A}, \mathcal{A})$ is $\mathrm{id}^{\mathsf{pFCD}(\mathfrak{A})}_{\mathcal{A}}$. (Note: this is defined only for meet-semilattices.)

To prove that it is really a category is trivial.

PROPOSITION 1568. For strongly separated and bounded $\mathfrak{A}$ and $\mathfrak{B}$ formula (28) is equivalent to each of the following:

1°. $\operatorname{dom} f \sqsubseteq \mathcal{A} \wedge \operatorname{im} f \sqsubseteq \mathcal{B}$;
2°. $f \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$.

PROOF. Because $\langle f \rangle x \sqsubseteq \operatorname{im} f$, $\langle f^{-1} \rangle y \sqsubseteq \operatorname{dom} f$, and theorem 1555. □