19.8. Funcoidal product of elements

DEFINITION 1552. *Funcoidal product* $\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ where $\mathcal{A} \in \mathfrak{A}$, $\mathcal{B} \in \mathfrak{B}$ and $\mathfrak{A}$ and $\mathfrak{B}$ are posets with least elements is a pointfree funcoid such that for every $\mathcal{X} \in \mathfrak{A}$, $\mathcal{Y} \in \mathfrak{B}$

$$\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle \mathcal{X} = \begin{cases} \mathcal{B} & \text{if } \mathcal{X} \not\asymp \mathcal{A}; \\ \bot^{\mathfrak{B}} & \text{if } \mathcal{X} \asymp \mathcal{A}; \end{cases} \quad \text{and} \quad \langle (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})^{-1} \rangle \mathcal{Y} = \begin{cases} \mathcal{A} & \text{if } \mathcal{Y} \not\asymp \mathcal{B}; \\ \bot^{\mathfrak{A}} & \text{if } \mathcal{Y} \asymp \mathcal{B}. \end{cases}$$

PROPOSITION 1553. $\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ is really a pointfree funcoid and

$$\mathcal{X} \left[\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \right] \mathcal{Y} \Leftrightarrow \mathcal{X} \not\asymp \mathcal{A} \wedge \mathcal{Y} \not\asymp \mathcal{B}.$$

PROOF. Obvious. □

PROPOSITION 1554. Let $\mathfrak{A}$ and $\mathfrak{B}$ be posets with least elements, $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$, $\mathcal{A} \in \mathfrak{A}$, $\mathcal{B} \in \mathfrak{B}$. Then

$$f \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \Rightarrow \operatorname{dom} f \sqsubseteq \mathcal{A} \wedge \operatorname{im} f \sqsubseteq \mathcal{B}.$$

PROOF. If $f \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ then $\operatorname{dom} f \sqsubseteq \operatorname{dom}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \sqsubseteq \mathcal{A}$, $\operatorname{im} f \sqsubseteq \operatorname{im}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \sqsubseteq \mathcal{B}$. □

THEOREM 1555. Let $\mathfrak{A}$ and $\mathfrak{B}$ be strongly separable bounded posets, $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$, $\mathcal{A} \in \mathfrak{A}$, $\mathcal{B} \in \mathfrak{B}$. Then

$$f \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \Leftrightarrow \operatorname{dom} f \sqsubseteq \mathcal{A} \wedge \operatorname{im} f \sqsubseteq \mathcal{B}.$$

PROOF. One direction is the proposition above. The other:

If $\operatorname{dom} f \sqsubseteq \mathcal{A} \wedge \operatorname{im} f \sqsubseteq \mathcal{B}$ then $\mathcal{X} \left[f \right] \mathcal{Y} \Rightarrow \mathcal{Y} \not\asymp \langle f \rangle \mathcal{X} \Rightarrow \mathcal{Y} \not\asymp \operatorname{im} f \Rightarrow \mathcal{Y} \not\asymp \mathcal{B}$ (strong separability used) and similarly $\mathcal{X} \left[f \right] \mathcal{Y} \Rightarrow \mathcal{X} \not\asymp \mathcal{A}$.

So $\left[f \right] \subseteq \left[\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \right]$ and thus using separability $f \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$. □

THEOREM 1556. Let $\mathfrak{A}$, $\mathfrak{B}$ be bounded separable meet-semilattices. For every $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ and $\mathcal{A} \in \mathfrak{A}$, $\mathcal{B} \in \mathfrak{B}$

$$f \sqcap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = \mathrm{id}^{\mathsf{pFCD}(\mathfrak{B})}_{\mathcal{B}} \circ f \circ \mathrm{id}^{\mathsf{pFCD}(\mathfrak{A})}_{\mathcal{A}}.$$

PROOF. $h \overset{\text{def}}{=} \mathrm{id}^{\mathsf{pFCD}(\mathfrak{B})}_{\mathcal{B}} \circ f \circ \mathrm{id}^{\mathsf{pFCD}(\mathfrak{A})}_{\mathcal{A}}$. For every $\mathcal{X} \in \mathfrak{A}$

$$\langle h \rangle \mathcal{X} = \left\langle \mathrm{id}^{\mathsf{pFCD}(\mathfrak{B})}_{\mathcal{B}} \right\rangle \langle f \rangle \left\langle \mathrm{id}^{\mathsf{pFCD}(\mathfrak{A})}_{\mathcal{A}} \right\rangle \mathcal{X} = \mathcal{B} \sqcap \langle f \rangle (\mathcal{A} \sqcap \mathcal{X})$$

and

$$\langle h^{-1} \rangle \mathcal{X} = \left\langle \mathrm{id}^{\mathsf{pFCD}(\mathfrak{A})}_{\mathcal{A}} \right\rangle \langle f^{-1} \rangle \left\langle \mathrm{id}^{\mathsf{pFCD}(\mathfrak{B})}_{\mathcal{B}} \right\rangle \mathcal{X} = \mathcal{A} \sqcap \langle f^{-1} \rangle (\mathcal{B} \sqcap \mathcal{X}).$$

From this, as easy to show, $h \sqsubseteq f$ and $h \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$. If $g \sqsubseteq f \wedge g \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ for a $g \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ then $\operatorname{dom} g \sqsubseteq \mathcal{A}$. $\mathfrak{A}$ and $\mathfrak{B}$ are are strongly separable by theorem 222. Thus by propositions 1537 we have:

$$\langle g \rangle \mathcal{X} = \langle g \rangle (\mathcal{X} \sqcap \operatorname{dom} g) = \langle g \rangle (\mathcal{X} \sqcap \mathcal{A}) = \mathcal{B} \sqcap \langle g \rangle (\mathcal{A} \sqcap \mathcal{X}) \sqsubseteq \mathcal{B} \sqcap \langle f \rangle (\mathcal{A} \sqcap \mathcal{X}) = \left\langle \mathrm{id}^{\mathsf{pFCD}(\mathfrak{B})}_{\mathcal{B}} \right\rangle \langle f \rangle \left\langle \mathrm{id}^{\mathsf{pFCD}(\mathfrak{A})}_{\mathcal{A}} \right\rangle \mathcal{X} = \langle h \rangle \mathcal{X},$$

and similarly $\langle g^{-1} \rangle \mathcal{Y} \sqsubseteq \langle h^{-1} \rangle \mathcal{Y}$. Thus $g \sqsubseteq h$.

So $h = f \sqcap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})$. □

COROLLARY 1557. Let $\mathfrak{A}$, $\mathfrak{B}$ be bounded separable meet-semilattices. For every $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ and $\mathcal{A} \in \mathfrak{A}$ we have $f|_{\mathcal{A}} = f \sqcap (\mathcal{A} \times^{\mathsf{FCD}} \top^{\mathfrak{B}})$.

PROOF. $f \sqcap (\mathcal{A} \times^{\mathsf{FCD}} \top^{\mathfrak{B}}) = \mathrm{id}^{\mathsf{pFCD}(\mathfrak{B})}_{\top^{\mathfrak{B}}} \circ f \circ \mathrm{id}^{\mathsf{pFCD}(\mathfrak{A})}_{\mathcal{A}} = f \circ \mathrm{id}^{\mathsf{pFCD}(\mathfrak{A})}_{\mathcal{A}} = f|_{\mathcal{A}}$. □