THEOREM 1549. Let f and g be pointfree funcoids and $\mathfrak{A} = \operatorname{Dst} f = \operatorname{Src} g$ be an atomic poset. Then for every $\mathcal{X} \in \operatorname{Src} f$ and $\mathcal{Z} \in \operatorname{Dst} g$

$$\mathcal{X} \, [g \circ f] \, \mathcal{Z} \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{A}} : (\mathcal{X} \, [f] \, y \wedge y \, [g] \, \mathcal{Z}).$$

PROOF.

$$\begin{aligned} \exists y \in \operatorname{atoms}^{\mathfrak{A}} : (\mathcal{X} \, [f] \, y \wedge y \, [g] \, \mathcal{Z}) \Leftrightarrow \\ \exists y \in \operatorname{atoms}^{\mathfrak{A}} : (\mathcal{Z} \not\asymp \langle g \rangle y \wedge y \not\asymp \langle f \rangle \mathcal{X}) \Leftrightarrow \\ \exists y \in \operatorname{atoms}^{\mathfrak{A}} : (y \not\asymp \langle g^{-1} \rangle \mathcal{Z} \wedge y \not\asymp \langle f \rangle \mathcal{X}) \Leftrightarrow \\ \langle g^{-1} \rangle \mathcal{Z} \not\asymp \langle f \rangle \mathcal{X} \Leftrightarrow \\ \mathcal{X} \, [g \circ f] \, \mathcal{Z}. \end{aligned}$$

□

THEOREM 1550. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be separable starrish join-semilattices and $\mathfrak{B}$ is atomic. Then:

1°. $f \circ (g \sqcup h) = f \circ g \sqcup f \circ h$ for $g, h \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ and $f \in \mathsf{pFCD}(\mathfrak{B}, \mathfrak{C})$.
2°. $(g \sqcup h) \circ f = g \circ f \sqcup h \circ f$ for $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ and $g, h \in \mathsf{pFCD}(\mathfrak{B}, \mathfrak{C})$.

PROOF. I will prove only the first equality because the other is analogous.
We can apply theorem 1526.
For every $\mathcal{X} \in \mathfrak{A}$, $\mathcal{Y} \in \mathfrak{C}$

$$\begin{aligned} \mathcal{X} \, [f \circ (g \sqcup h)] \, \mathcal{Z} \Leftrightarrow \\ \exists y \in \operatorname{atoms}^{\mathfrak{B}} : (\mathcal{X} \, [g \sqcup h] \, y \wedge y \, [f] \, \mathcal{Z}) \Leftrightarrow \\ \exists y \in \operatorname{atoms}^{\mathfrak{B}} : ((\mathcal{X} \, [g] \, y \vee \mathcal{X} \, [h] \, y) \wedge y \, [f] \, \mathcal{Z}) \Leftrightarrow \\ \exists y \in \operatorname{atoms}^{\mathfrak{B}} : ((\mathcal{X} \, [g] \, y \wedge y \, [f] \, \mathcal{Z}) \vee (\mathcal{X} \, [h] \, y \wedge y \, [f] \, \mathcal{Z})) \Leftrightarrow \\ \exists y \in \operatorname{atoms}^{\mathfrak{B}} : (\mathcal{X} \, [g] \, y \wedge y \, [f] \, \mathcal{Z}) \vee \exists y \in \operatorname{atoms}^{\mathfrak{B}} : (\mathcal{X} \, [h] \, y \wedge y \, [f] \, \mathcal{Z}) \Leftrightarrow \\ \mathcal{X} \, [f \circ g] \, \mathcal{Z} \vee \mathcal{X} \, [f \circ h] \, \mathcal{Z} \Leftrightarrow \\ \mathcal{X} \, [f \circ g \sqcup f \circ h] \, \mathcal{Z}. \end{aligned}$$

Thus $f \circ (g \sqcup h) = f \circ g \sqcup f \circ h$ by theorem 1495. □

THEOREM 1551. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be posets of filters over some boolean lattices, $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$, $g \in \mathsf{pFCD}(\mathfrak{B}, \mathfrak{C})$, $h \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{C})$. Then

$$g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{-1}.$$

PROOF.

$$\begin{aligned} g \circ f \not\asymp h \Leftrightarrow \\ \exists a \in \operatorname{atoms}^{\mathfrak{A}}, c \in \operatorname{atoms}^{\mathfrak{C}} : a \, [(g \circ f) \sqcap h] \, c \Leftrightarrow \\ \exists a \in \operatorname{atoms}^{\mathfrak{A}}, c \in \operatorname{atoms}^{\mathfrak{C}} : (a \, [g \circ f] \, c \wedge a \, [h] \, c) \Leftrightarrow \\ \exists a \in \operatorname{atoms}^{\mathfrak{A}}, b \in \operatorname{atoms}^{\mathfrak{B}}, c \in \operatorname{atoms}^{\mathfrak{C}} : (a \, [f] \, b \wedge b \, [g] \, c \wedge a \, [h] \, c) \Leftrightarrow \\ \exists b \in \operatorname{atoms}^{\mathfrak{B}}, c \in \operatorname{atoms}^{\mathfrak{C}} : (b \, [g] \, c \wedge b \, \left[h \circ f^{-1}\right] \, c) \Leftrightarrow \\ \exists b \in \operatorname{atoms}^{\mathfrak{B}}, c \in \operatorname{atoms}^{\mathfrak{C}} : b \, \left[g \sqcap (h \circ f^{-1})\right] \, c \Leftrightarrow \\ g \not\asymp h \circ f^{-1}. \end{aligned}$$

□