19.6. Specifying funcoids by functions or relations on atomic filters

THEOREM 1542. Let $\mathfrak{A}$ be an atomic poset and $(\mathfrak{B}, \mathfrak{Z}_1)$ is a primary filtrator over a boolean lattice. Then for every $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ and $\mathcal{X} \in \mathfrak{A}$ we have

$$\langle f\rangle\mathcal{X} = \bigsqcup^{\mathfrak{B}} \langle\langle f\rangle\rangle^* \operatorname{atoms}^{\mathfrak{A}} \mathcal{X}.$$

PROOF. For every $Y \in \mathfrak{Z}_1$ we have

$$Y \not\asymp^{\mathfrak{B}} \langle f\rangle\mathcal{X} \Leftrightarrow \mathcal{X} \not\asymp^{\mathfrak{A}} \langle f^{-1}\rangle Y \Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{A}} \mathcal{X} : x \not\asymp^{\mathfrak{A}} \langle f^{-1}\rangle Y \Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{A}} \mathcal{X} : Y \not\asymp^{\mathfrak{B}} \langle f\rangle x.$$

Thus $\partial\langle f\rangle\mathcal{X} = \bigcup\langle\partial\rangle^*\langle\langle f\rangle\rangle^* \operatorname{atoms}^{\mathfrak{A}} \mathcal{X} = \partial\bigsqcup^{\mathfrak{B}}\langle\langle f\rangle\rangle^* \operatorname{atoms}^{\mathfrak{A}} \mathcal{X}$ (used corollary 566). Consequently $\langle f\rangle\mathcal{X} = \bigsqcup^{\mathfrak{B}}\langle\langle f\rangle\rangle^* \operatorname{atoms}^{\mathfrak{A}} \mathcal{X}$ by the corollary 565. □

PROPOSITION 1543. Let f be a pointfree funcoid. Then for every $\mathcal{X} \in \operatorname{Src} f$ and $\mathcal{Y} \in \operatorname{Dst} f$

1°. $\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \exists x \in \operatorname{atoms} \mathcal{X} : x [f] \mathcal{Y}$ if $\operatorname{Src} f$ is an atomic poset.
2°. $\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \exists y \in \operatorname{atoms} \mathcal{Y} : \mathcal{X} [f] y$ if $\operatorname{Dst} f$ is an atomic poset.

PROOF. I will prove only the second as the first is similar.

If $\mathcal{X} [f] \mathcal{Y}$, then $\mathcal{Y} \not\asymp \langle f\rangle\mathcal{X}$, consequently exists $y \in \operatorname{atoms} \mathcal{Y}$ such that $y \not\asymp \langle f\rangle\mathcal{X}$, $\mathcal{X} [f] y$. The reverse is obvious. □

COROLLARY 1544. If f is a pointfree funcoid with both source and destination being atomic posets, then for every $\mathcal{X} \in \operatorname{Src} f$ and $\mathcal{Y} \in \operatorname{Dst} f$

$$\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \exists x \in \operatorname{atoms} \mathcal{X}, y \in \operatorname{atoms} \mathcal{Y} : x [f] y.$$

PROOF. Apply the theorem twice. □

COROLLARY 1545. If $\mathfrak{A}$ is a separable atomic poset and $\mathfrak{B}$ is a separable poset then $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ is determined by the values of $\langle f\rangle X$ for $X \in \operatorname{atoms}^{\mathfrak{A}}$.

PROOF.

$$y \not\asymp \langle f\rangle x \Leftrightarrow x \not\asymp \langle f^{-1}\rangle y \Leftrightarrow \exists X \in \operatorname{atoms} x : X \not\asymp \langle f^{-1}\rangle y \Leftrightarrow \exists X \in \operatorname{atoms} x : y \not\asymp \langle f\rangle X.$$

Thus by separability of $\mathfrak{B}$ we have $\langle f\rangle$ is determined by $\langle f\rangle X$ for $X \in \operatorname{atoms} x$.

By separability of $\mathfrak{A}$ we infer that f can be restored from $\langle f\rangle$ (theorem 1495). □

THEOREM 1546. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices.

1°. A function $\alpha \in \mathfrak{B}^{\operatorname{atoms}^{\mathfrak{A}}}$ such that (for every $a \in \operatorname{atoms}^{\mathfrak{A}}$)

$$\alpha a \sqsubseteq \bigsqcap\left\langle\bigsqcup \circ\langle\alpha\rangle^* \circ \operatorname{atoms}^{\mathfrak{A}}\right\rangle^* \operatorname{up}^{\mathfrak{Z}_0} a \tag{24}$$

can be continued to the function $\langle f\rangle$ for a unique $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$;

$$\langle f\rangle\mathcal{X} = \bigsqcup\langle\alpha\rangle^* \operatorname{atoms}^{\mathfrak{A}} \mathcal{X} \tag{25}$$

for every $\mathcal{X} \in \mathfrak{A}$.

2°. A relation $\delta \in \mathscr{P}(\operatorname{atoms}^{\mathfrak{A}} \times \operatorname{atoms}^{\mathfrak{B}})$ such that (for every $a \in \operatorname{atoms}^{\mathfrak{A}}$, $b \in \operatorname{atoms}^{\mathfrak{B}}$)

$$\forall X \in \operatorname{up}^{\mathfrak{Z}_0} a, Y \in \operatorname{up}^{\mathfrak{Z}_1} b \exists x \in \operatorname{atoms}^{\mathfrak{A}} X, y \in \operatorname{atoms}^{\mathfrak{B}} Y : x \mathrel{\delta} y \Rightarrow a \mathrel{\delta} b \tag{26}$$

can be continued to the relation $[f]$ for a unique $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$;

$$\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \exists x \in \operatorname{atoms} \mathcal{X}, y \in \operatorname{atoms} \mathcal{Y} : x \mathrel{\delta} y \tag{27}$$

for every $\mathcal{X} \in \mathfrak{A}$, $\mathcal{Y} \in \mathfrak{B}$.