So $h = (\mathfrak{A}, \mathfrak{B}, \alpha, \beta)$ is a pointfree funcoid. Obviously $h \sqsupseteq f$ and $h \sqsupseteq g$. If $p \sqsupseteq f$ and $p \sqsupseteq g$ for some $p \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ then $\langle p\rangle x \sqsupseteq \langle f\rangle x \sqcup \langle g\rangle x = \langle h\rangle x$ and $\left\langle p^{-1}\right\rangle y \sqsupseteq \left\langle f^{-1}\right\rangle y \sqcup \left\langle g^{-1}\right\rangle y = \left\langle h^{-1}\right\rangle y$ that is $p \sqsupseteq h$. So $f \sqcup g = h$.

2°.

$$\begin{aligned} x\ [f \sqcup g]\ y &\Leftrightarrow \\ y \not\asymp \langle f \sqcup g\rangle x &\Leftrightarrow \\ y \not\asymp \langle f\rangle x \sqcup \langle g\rangle x &\Leftrightarrow \\ y \not\asymp \langle f\rangle x \vee y \not\asymp \langle g\rangle x &\Leftrightarrow \\ x\ [f]\ y \vee x\ [g]\ y & \end{aligned}$$

for every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$.

19.5. Domain and range of a pointfree funcoid

DEFINITION 1527. Let $\mathfrak{A}$ be a poset. The *identity pointfree funcoid* $1^{\mathsf{pFCD}}_{\mathfrak{A}} = (\mathfrak{A}, \mathfrak{A}, \mathrm{id}_{\mathfrak{A}}, \mathrm{id}_{\mathfrak{A}})$.

It is trivial that identity funcoid is really a pointfree funcoid.

Let now $\mathfrak{A}$ be a meet-semilattice.

DEFINITION 1528. Let $a \in \mathfrak{A}$. The *restricted identity pointfree funcoid* $\mathrm{id}^{\mathsf{pFCD}(\mathfrak{A})}_a = (\mathfrak{A}, \mathfrak{A}, a \sqcap^{\mathfrak{A}}, a \sqcap^{\mathfrak{A}})$.

PROPOSITION 1529. The restricted pointfree funcoid is a pointfree funcoid.

PROOF. We need to prove that $(a \sqcap^{\mathfrak{A}} x) \not\asymp^{\mathfrak{A}} y \Leftrightarrow (a \sqcap^{\mathfrak{A}} y) \not\asymp^{\mathfrak{A}} x$ what is obvious. □

OBVIOUS 1530. $(\mathrm{id}^{\mathsf{pFCD}(\mathfrak{A})}_a)^{-1} = \mathrm{id}^{\mathsf{pFCD}(\mathfrak{A})}_a$.

OBVIOUS 1531. $x \left[\mathrm{id}^{\mathsf{pFCD}(\mathfrak{A})}_a\right] y \Leftrightarrow a \not\asymp^{\mathfrak{A}} x \sqcap^{\mathfrak{A}} y$ for every $x, y \in \mathfrak{A}$.

DEFINITION 1532. I will define *restricting* of a pointfree funcoid f to an element $a \in \operatorname{Src} f$ by the formula $f|_a \overset{\text{def}}{=} f \circ \mathrm{id}^{\mathsf{pFCD}(\operatorname{Src} f)}_a$.

DEFINITION 1533. Let f be a pointfree funcoid whose source is a set with greatest element. *Image* of f will be defined by the formula $\operatorname{im} f = \langle f\rangle\top$.

PROPOSITION 1534. $\operatorname{im} f \sqsupseteq \langle f\rangle x$ for every $x \in \operatorname{Src} f$ whenever $\operatorname{Dst} f$ is a strongly separable poset with greatest element.

PROOF. $\langle f\rangle\top$ is greater than every $\langle f\rangle x$ (where $x \in \operatorname{Src} f$) by proposition 1497. □

DEFINITION 1535. *Domain* of a pointfree funcoid f is defined by the formula $\operatorname{dom} f = \operatorname{im} f^{-1}$.

PROPOSITION 1536. $\langle f\rangle \operatorname{dom} f = \operatorname{im} f$ if f is a pointfree funcoid and $\operatorname{Src} f$ is a strongly separable poset with greatest element and $\operatorname{Dst} f$ is a separable poset with greatest element.

PROOF. For every $y \in \operatorname{Dst} f$

$$y \not\asymp \langle f\rangle \operatorname{dom} f \Leftrightarrow \operatorname{dom} f \not\asymp \left\langle f^{-1}\right\rangle y \Leftrightarrow \left\langle f^{-1}\right\rangle\top \not\asymp \left\langle f^{-1}\right\rangle y \Leftrightarrow$$
$$\text{(by strong separability of } \operatorname{Src} f)$$
$$\left\langle f^{-1}\right\rangle y \text{ is not least} \Leftrightarrow \top \not\asymp \left\langle f^{-1}\right\rangle y \Leftrightarrow y \not\asymp \langle f\rangle\top \Leftrightarrow y \not\asymp \operatorname{im} f.$$

So $\langle f\rangle \operatorname{dom} f = \operatorname{im} f$ by separability of $\operatorname{Dst} f$. □