2°. $\alpha X \overset{\text{def}}{=} \bigsqcup_{f\in R}\langle f\rangle X$ (by corollary 515 all joins on $\mathfrak{B}$ exist). We have $\alpha\bot^{\mathfrak{A}} = \bot^{\mathfrak{B}}$;

$$\begin{aligned}
&\alpha(I \sqcup^{\mathfrak{Z}_0} J) = \\
&\bigsqcup\left\{\frac{\langle f\rangle(I \sqcup^{\mathfrak{Z}_0} J)}{f\in R}\right\} = \\
&\bigsqcup\left\{\frac{\langle f\rangle(I \sqcup^{\mathfrak{A}} J)}{f\in R}\right\} = \\
&\bigsqcup\left\{\frac{\langle f\rangle I \sqcup^{\mathfrak{B}} \langle f\rangle J}{f\in R}\right\} = \\
&\bigsqcup\left\{\frac{\langle f\rangle I}{f\in R}\right\} \sqcup^{\mathfrak{B}} \bigsqcup\left\{\frac{\langle f\rangle J}{f\in R}\right\} = \\
&\alpha I \sqcup^{\mathfrak{B}} \alpha J
\end{aligned}$$

(used theorem 1498). By theorem 1510 the function α can be continued to $\langle h\rangle$ for an $h \in \mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$. Obviously

$$\forall f \in R : h \sqsupseteq f. \tag{23}$$

And h is the least element of $\mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$ for which the condition (23) holds. So $h = \bigsqcup R$.

1°.

$$\begin{aligned}
&X \left[\bigsqcup R\right] Y \Leftrightarrow \\
&Y \sqcap^{\mathfrak{B}} \left\langle\bigsqcup R\right\rangle X \neq \bot^{\mathfrak{B}} \Leftrightarrow \\
&Y \sqcap^{\mathfrak{B}} \bigsqcup\left\{\frac{\langle f\rangle X}{f\in R}\right\} \neq \bot^{\mathfrak{B}} \Leftrightarrow \\
&\exists f \in R : Y \sqcap^{\mathfrak{B}} \langle f\rangle X \neq \bot^{\mathfrak{B}} \Leftrightarrow \\
&\exists f \in R : X\ [f]\ Y
\end{aligned}$$

(used theorem 607).

□

COROLLARY 1525. If $(\mathfrak{A},\mathfrak{Z}_0)$ and $(\mathfrak{B},\mathfrak{Z}_1)$ are primary filtrators over boolean lattices then $\mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$ is a complete lattice.

PROOF. Apply [27]. □

THEOREM 1526. Let $\mathfrak{A}$ and $\mathfrak{B}$ be starrish join-semilattices. Then for $f, g \in \mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$:

1°. $\langle f \sqcup g\rangle x = \langle f\rangle x \sqcup \langle g\rangle x$ for every $x \in \mathfrak{A}$;
2°. $[f \sqcup g]=[f] \cup [g]$.

PROOF. □

1°. Let $\alpha\mathcal{X} \overset{\text{def}}{=} \langle f\rangle x \sqcup \langle g\rangle x$; $\beta\mathcal{Y} \overset{\text{def}}{=} \langle f^{-1}\rangle y \sqcup \langle g^{-1}\rangle y$ for every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$. Then

$$\begin{aligned}
&y \not\asymp^{\mathfrak{B}} \alpha x \Leftrightarrow \\
&y \not\asymp \langle f\rangle x \vee y \not\asymp \langle g\rangle x \Leftrightarrow \\
&x \not\asymp \langle f^{-1}\rangle y \vee x \not\asymp \langle g^{-1}\rangle y \Leftrightarrow \\
&x \not\asymp \langle f^{-1}\rangle y \sqcup \langle g^{-1}\rangle y \Leftrightarrow \\
&x \not\asymp \beta y.
\end{aligned}$$