Taking into account properties of generalized filter bases:

$$\langle f\rangle \bigsqcap^{\operatorname{Src} f} S =$$

$$\bigsqcap^{\operatorname{Dst} f} \langle\langle f\rangle\rangle^* \operatorname{up} \bigsqcap S =$$

$$\bigsqcap^{\operatorname{Dst} f} \langle\langle f\rangle\rangle^* \left\{ \frac{X}{\exists \mathcal{P} \in S : X \in \operatorname{up} \mathcal{P}} \right\} =$$

$$\bigsqcap^{\operatorname{Dst} f} \left\{ \frac{\langle f\rangle^* X}{\exists \mathcal{P} \in S : X \in \operatorname{up} \mathcal{P}} \right\} \sqsupseteq \text{ (because } \operatorname{Dst} f \text{ is a strongly separable poset)}$$

$$\bigsqcap^{\operatorname{Dst} f} \left\{ \frac{\langle f\rangle \mathcal{P}}{\mathcal{P} \in S} \right\} =$$

$$\bigsqcap^{\operatorname{Dst} f} \langle\langle f\rangle\rangle^* S.$$

□

Proposition 1513. $\mathcal{X}\ [f] \bigsqcap S \Leftrightarrow \exists \mathcal{Y} \in S : \mathcal{X}\ [f]\ \mathcal{Y}$ if f is a pointfree funcoid, $\operatorname{Dst} f$ is a meet-semilattice with least element and S is a generalized filter base on $\operatorname{Dst} f$.

Proof.

$$\mathcal{X}\ [f] \bigsqcap S \Leftrightarrow \bigsqcap S \sqcap \langle f\rangle \mathcal{X} \neq \bot \Leftrightarrow \bigsqcap \langle\langle f\rangle \mathcal{X} \sqcap\rangle^* S \neq \bot \Leftrightarrow$$
$$\text{(by properties of generalized filter bases)} \Leftrightarrow$$
$$\exists \mathcal{Y} \in \langle\langle f\rangle \mathcal{X} \sqcap\rangle^* S : \mathcal{Y} \neq \bot \Leftrightarrow \exists \mathcal{Y} \in S : \langle f\rangle \mathcal{X} \sqcap \mathcal{Y} \neq \bot \Leftrightarrow \exists \mathcal{Y} \in S : \mathcal{X}\ [f]\ \mathcal{Y}.$$

□

Theorem 1514. A function $\varphi : \mathfrak{A} \to \mathfrak{B}$, where $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ are primary filtrators over boolean lattices, preserves finite joins (including nullary joins) and filtered meets iff there exists a pointfree funcoid f such that $\langle f\rangle = \varphi$.

Proof. Backward implication follows from above.

Let $\psi = \varphi|_{\mathfrak{Z}_0}$. Then ψ preserves bottom element and binary joins. Thus there exists a funcoid f such that $\langle f\rangle^* = \psi$.

It remains to prove that $\langle f\rangle = \varphi$.

Really, $\langle f\rangle \mathcal{X} = \bigsqcap \langle\langle f\rangle\rangle^* \operatorname{up} \mathcal{X} = \bigsqcap \langle\psi\rangle^* \operatorname{up} \mathcal{X} = \bigsqcap \langle\varphi\rangle^* \operatorname{up} \mathcal{X} = \varphi \bigsqcap \operatorname{up} \mathcal{X} = \varphi \mathcal{X}$ for every $\mathcal{X} \in \mathscr{F}(\operatorname{Src} f)$. □

Corollary 1515. Pointfree funcoids f from a lattice $\mathfrak{A}$ of fitlters on a boolean lattice to a lattice $\mathfrak{B}$ of fitlters on a boolean lattice bijectively correspond by the formula $\langle f\rangle = \varphi$ to functions $\varphi : \mathfrak{A} \to \mathfrak{B}$ preserving finite joins and filtered meets.

Theorem 1516. The set of pointfree funcoids between sets of filters on boolean lattices is a co-frame.

Proof. Theorems 1510 and 530. □

19.4. The order of pointfree funcoids

Definition 1517. The order of pointfree funcoids $\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ is defined by the formula:

$$f \sqsubseteq g \Leftrightarrow \forall x \in \mathfrak{A} : \langle f\rangle x \sqsubseteq \langle g\rangle x \wedge \forall y \in \mathfrak{B} : \left\langle f^{-1}\right\rangle y \sqsubseteq \left\langle g^{-1}\right\rangle y.$$

Proposition 1518. It is really a partial order on the set $\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$.