19.3. Pointfree funcoid as continuation

PROPOSITION 1507. Let f be a pointfree funcoid. Then for every $x \in \operatorname{Src} f$, $y \in \operatorname{Dst} f$ we have

1°. If $(\operatorname{Src} f, \mathfrak{Z})$ is a filtrator with separable core then $x \,[f]\, y \Leftrightarrow \forall X \in \operatorname{up}^{\mathfrak{Z}} x : X \,[f]\, y$.

2°. If $(\operatorname{Dst} f, \mathfrak{Z})$ is a filtrator with separable core then $x \,[f]\, y \Leftrightarrow \forall Y \in \operatorname{up}^{\mathfrak{Z}} y : x \,[f]\, Y$.

PROOF. We will prove only the second because the first is similar.

$$x \,[f]\, y \Leftrightarrow y \not\asymp^{\operatorname{Dst} f} \langle f \rangle x \Leftrightarrow \forall Y \in \operatorname{up}^{\mathfrak{Z}} y : Y \not\asymp \langle f \rangle x \Leftrightarrow \forall Y \in \operatorname{up}^{\mathfrak{Z}} y : x \,[f]\, Y.$$

□

COROLLARY 1508. Let f be a pointfree funcoid and $(\operatorname{Src} f, \mathfrak{Z}_0)$, $(\operatorname{Dst} f, \mathfrak{Z}_1)$ be filtrators with separable core. Then

$$x \,[f]\, y \Leftrightarrow \forall X \in \operatorname{up}^{\mathfrak{Z}_0} x, Y \in \operatorname{up}^{\mathfrak{Z}_1} y : X \,[f]\, Y.$$

PROOF. Apply the proposition twice. □

THEOREM 1509. Let f be a pointfree funcoid. Let $(\operatorname{Src} f, \mathfrak{Z}_0)$ be a binarily meet-closed filtrator with separable core which is a meet-semilattice and $\forall x \in \operatorname{Src} f : \operatorname{up}^{\mathfrak{Z}_0} x \neq \emptyset$ and $(\operatorname{Dst} f, \mathfrak{Z}_1)$ be a primary filtrator over a boolean lattice.

$$\langle f \rangle x = \bigsqcap^{\operatorname{Dst} f} \langle \langle f \rangle \rangle^* \operatorname{up}^{\mathfrak{Z}_0} x.$$

PROOF. By the previous proposition for every $y \in \operatorname{Dst} f$:

$$y \not\asymp^{\operatorname{Dst} f} \langle f \rangle x \Leftrightarrow x \,[f]\, y \Leftrightarrow \forall X \in \operatorname{up}^{\mathfrak{Z}_0} x : X \,[f]\, y \Leftrightarrow \forall X \in \operatorname{up}^{\mathfrak{Z}_0} x : y \not\asymp^{\operatorname{Dst} f} \langle f \rangle X.$$

Let's denote $W = \left\{ \frac{y \sqcap^{\operatorname{Dst} f} \langle f \rangle X}{X \in \operatorname{up}^{\mathfrak{Z}_0} x} \right\}$. We will prove that W is a generalized filter base over $\mathfrak{Z}_1$. To prove this enough to show that $V = \left\{ \frac{\langle f \rangle X}{X \in \operatorname{up}^{\mathfrak{Z}_0} x} \right\}$ is a generalized filter base.

Let $\mathcal{P}, \mathcal{Q} \in V$. Then $\mathcal{P} = \langle f \rangle A$, $\mathcal{Q} = \langle f \rangle B$ where $A, B \in \operatorname{up}^{\mathfrak{Z}_0} x$; $A \sqcap^{\mathfrak{Z}_0} B \in \operatorname{up}^{\mathfrak{Z}_0} x$ (used the fact that it is a binarily meet-closed and theorem 532) and $\mathcal{R} \sqsubseteq \mathcal{P} \sqcap^{\operatorname{Dst} f} \mathcal{Q}$ for $\mathcal{R} = \langle f \rangle (A \sqcap^{\mathfrak{Z}_0} B) \in V$ because $\operatorname{Dst} f$ is strongly separable by proposition 576. So V is a generalized filter base and thus W is a generalized filter base.

$\bot^{\operatorname{Dst} f} \notin W \Leftrightarrow \bot^{\operatorname{Dst} f} \notin \bigsqcap^{\operatorname{Dst} f} W$ by theorem 569. That is

$$\forall X \in \operatorname{up}^{\mathfrak{Z}_0} x : y \sqcap^{\operatorname{Dst} f} \langle f \rangle X \neq \bot^{\operatorname{Dst} f} \Leftrightarrow y \sqcap^{\operatorname{Dst} f} \bigsqcap^{\operatorname{Dst} f} \langle \langle f \rangle \rangle^* \operatorname{up}^{\mathfrak{Z}_0} x \neq \bot^{\operatorname{Dst} f}.$$

Comparing with the above,

$$y \sqcap^{\operatorname{Dst} f} \langle f \rangle x \neq \bot^{\operatorname{Dst} f} \Leftrightarrow y \sqcap^{\operatorname{Dst} f} \bigsqcap^{\operatorname{Dst} f} \langle \langle f \rangle \rangle^* \operatorname{up}^{\mathfrak{Z}_0} x \neq \bot^{\operatorname{Dst} f}.$$

So $\langle f \rangle x = \bigsqcap^{\operatorname{Dst} f} \langle \langle f \rangle \rangle^* \operatorname{up}^{\mathfrak{Z}_0} x$ because $\operatorname{Dst} f$ is separable (proposition 576 and the fact that $\mathfrak{Z}_1$ is a boolean lattice). □

THEOREM 1510. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices.

1°. A function $\alpha \in \mathfrak{B}^{\mathfrak{Z}_0}$ conforming to the formulas (for every $I, J \in \mathfrak{Z}_0$)

$$\alpha \bot^{\mathfrak{Z}_0} = \bot^{\mathfrak{B}}, \quad \alpha (I \sqcup J) = \alpha I \sqcup \alpha J$$