Thus $\langle f\rangle(i\sqcup j)=\langle f\rangle i\sqcup\langle f\rangle j$ by separability. □

PROPOSITION 1499. Let f be a pointfree funcoid. Then:

1°. $k\ [f]\ i\sqcup j\Leftrightarrow k\ [f]\ i\vee k\ [f]\ j$ for every $i,j\in\operatorname{Dst}f$, $k\in\operatorname{Src}f$ if $\operatorname{Dst}f$ is a starrish join-semilattice.
2°. $i\sqcup j\ [f]\ k\Leftrightarrow i\ [f]\ k\vee j\ [f]\ k$ for every $i,j\in\operatorname{Src}f$, $k\in\operatorname{Dst}f$ if $\operatorname{Src}f$ is a starrish join-semilattice.

PROOF.
1°. $k\ [f]\ i\sqcup j\Leftrightarrow i\sqcup j\not\asymp\langle f\rangle k\Leftrightarrow i\not\asymp\langle f\rangle k\vee j\not\asymp\langle f\rangle k\Leftrightarrow k\ [f]\ i\vee k\ [f]\ j$.
2°. Similar.

□

19.2. Composition of pointfree funcoids

DEFINITION 1500. *Composition* of pointfree funcoids is defined by the formula

$$(\mathfrak{B},\mathfrak{C},\alpha_2,\beta_2)\circ(\mathfrak{A},\mathfrak{B},\alpha_1,\beta_1)=(\mathfrak{A},\mathfrak{C},\alpha_2\circ\alpha_1,\beta_1\circ\beta_2).$$

DEFINITION 1501. I will call funcoids f and g *composable* when $\operatorname{Dst}f=\operatorname{Src}g$.

PROPOSITION 1502. If f, g are composable pointfree funcoids then $g\circ f$ is pointfree funcoid.

PROOF. Let $f=(\mathfrak{A},\mathfrak{B},\alpha_1,\beta_1)$, $g=(\mathfrak{B},\mathfrak{C},\alpha_2,\beta_2)$. For every $x,y\in\mathfrak{A}$ we have

$$y\not\asymp(\alpha_2\circ\alpha_1)x\Leftrightarrow y\not\asymp\alpha_2\alpha_1x\Leftrightarrow\alpha_1x\not\asymp\beta_2y\Leftrightarrow x\not\asymp\beta_1\beta_2y\Leftrightarrow x\not\asymp(\beta_1\circ\beta_2)y.$$

So $(\mathfrak{A},\mathfrak{C},\alpha_2\circ\alpha_1,\beta_1\circ\beta_2)$ is a pointfree funcoid. □

OBVIOUS 1503. $\langle g\circ f\rangle=\langle g\rangle\circ\langle f\rangle$ for every composable pointfree funcoids f and g.

THEOREM 1504. $(g\circ f)^{-1}=f^{-1}\circ g^{-1}$ for every composable pointfree funcoids f and g.

PROOF.

$$\left\langle(g\circ f)^{-1}\right\rangle=\left\langle f^{-1}\right\rangle\circ\left\langle g^{-1}\right\rangle=\left\langle f^{-1}\circ g^{-1}\right\rangle;$$
$$\left\langle((g\circ f)^{-1})^{-1}\right\rangle=\langle g\circ f\rangle=\left\langle(f^{-1}\circ g^{-1})^{-1}\right\rangle.$$

□

PROPOSITION 1505. $(h\circ g)\circ f=h\circ(g\circ f)$ for every composable pointfree funcoids f, g, h.

PROOF. $\langle(h\circ g)\circ f\rangle=\langle h\circ g\rangle\circ\langle f\rangle=\langle h\rangle\circ\langle g\rangle\circ\langle f\rangle=\langle h\rangle\circ\langle g\circ f\rangle=\langle h\circ(g\circ f)\rangle$;

$$\left\langle((h\circ g)\circ f)^{-1}\right\rangle=\left\langle f^{-1}\circ(h\circ g)^{-1}\right\rangle=\left\langle f^{-1}\circ g^{-1}\circ h^{-1}\right\rangle=\left\langle(g\circ f)^{-1}\circ h^{-1}\right\rangle=\left\langle(h\circ(g\circ f))^{-1}\right\rangle.$$

□

EXERCISE 1506. Generalize section 7.4 for pointfree funcoids.