18.2. Relationships between convergence and continuity

LEMMA 1457. Let μ, ν be endofuncoids, $f \in \mathsf{FCD}(\operatorname{Ob}\mu, \operatorname{Ob}\nu)$, $\mathcal{A} \in \mathscr{F}(\operatorname{Ob}\mu)$, $\operatorname{Src} f = \operatorname{Ob}\mu$, $\operatorname{Dst} f = \operatorname{Ob}\nu$. If $f \in \mathrm{C}(\mu|_{\mathcal{A}}, \nu)$ then

$$f|_{\langle\mu\rangle\mathcal{A}} \xrightarrow{\nu} \langle f\rangle\mathcal{A} \Leftrightarrow \langle f \circ \mu|_{\mathcal{A}}\rangle\mathcal{A} \sqsubseteq \langle \nu \circ f\rangle\mathcal{A}.$$

PROOF.

$$f|_{\langle\mu\rangle\mathcal{A}} \xrightarrow{\nu} \langle f\rangle\mathcal{A} \Leftrightarrow \operatorname{im} f|_{\langle\mu\rangle\mathcal{A}} \sqsubseteq \langle\nu\rangle\langle f\rangle\mathcal{A} \Leftrightarrow \langle f\rangle\langle\mu\rangle\mathcal{A} \sqsubseteq \langle\nu\rangle\langle f\rangle\mathcal{A} \Leftrightarrow \langle f \circ \mu\rangle\mathcal{A} \sqsubseteq \langle \nu \circ f\rangle\mathcal{A} \Leftrightarrow \langle f \circ \mu|_{\mathcal{A}}\rangle\mathcal{A} \sqsubseteq \langle \nu \circ f\rangle\mathcal{A}.$$

□

THEOREM 1458. Let μ, ν be endofuncoids, $f \in \mathsf{FCD}(\operatorname{Ob}\mu, \operatorname{Ob}\nu)$, $\mathcal{A} \in \mathscr{F}(\operatorname{Ob}\mu)$, $\operatorname{Src} f = \operatorname{Ob}\mu$, $\operatorname{Dst} f = \operatorname{Ob}\nu$. If $f \in \mathrm{C}(\mu|_{\mathcal{A}}, \nu)$ then $f|_{\langle\mu\rangle\mathcal{A}} \xrightarrow{\nu} \langle f\rangle\mathcal{A}$.

PROOF.

$$f|_{\langle\mu\rangle\mathcal{A}} \xrightarrow{\nu} \langle f\rangle\mathcal{A} \Leftrightarrow (\text{by the lemma}) \Leftrightarrow \langle f \circ \mu|_{\mathcal{A}}\rangle\mathcal{A} \sqsubseteq \langle \nu \circ f\rangle\mathcal{A} \Leftarrow f \circ \mu|_{\mathcal{A}} \sqsubseteq \nu \circ f \Leftrightarrow f \in \mathrm{C}(\mu|_{\mathcal{A}}, \nu).$$

□

COROLLARY 1459. Let μ, ν be endofuncoids, $f \in \mathsf{FCD}(\operatorname{Ob}\mu, \operatorname{Ob}\nu)$, $\mathcal{A} \in \mathscr{F}(\operatorname{Ob}\mu)$, $\operatorname{Src} f = \operatorname{Ob}\mu$, $\operatorname{Dst} f = \operatorname{Ob}\nu$. If $f \in \mathrm{C}(\mu, \nu)$ then $f|_{\langle\mu\rangle\mathcal{A}} \xrightarrow{\nu} \langle f\rangle\mathcal{A}$.

THEOREM 1460. Let μ, ν be endofuncoids, $f \in \mathsf{FCD}(\operatorname{Ob}\mu, \operatorname{Ob}\nu)$, $\mathcal{A} \in \mathscr{F}(\operatorname{Ob}\mu)$ be an ultrafilter, $\operatorname{Src} f = \operatorname{Ob}\mu$, $\operatorname{Dst} f = \operatorname{Ob}\nu$. $f \in \mathrm{C}(\mu|_{\mathcal{A}}, \nu)$ iff $f|_{\langle\mu\rangle\mathcal{A}} \xrightarrow{\nu} \langle f\rangle\mathcal{A}$.

PROOF.

$$f|_{\langle\mu\rangle\mathcal{A}} \xrightarrow{\nu} \langle f\rangle\mathcal{A} \Leftrightarrow (\text{by the lemma}) \Leftrightarrow \langle f \circ \mu|_{\mathcal{A}}\rangle\mathcal{A} \sqsubseteq \langle \nu \circ f\rangle\mathcal{A} \Leftrightarrow$$
$$(\text{used the fact that } \mathcal{A} \text{ is an ultrafilter})$$
$$f \circ \mu|_{\mathcal{A}} \sqsubseteq \nu \circ f|_{\mathcal{A}} \Leftrightarrow f \circ \mu|_{\mathcal{A}} \sqsubseteq \nu \circ f \Leftrightarrow f \in \mathrm{C}(\mu|_{\mathcal{A}}, \nu).$$

□

18.3. Convergence of join

PROPOSITION 1461. $\bigsqcup S \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \forall \mathcal{F} \in S : \mathcal{F} \xrightarrow{\mu} \mathcal{A}$ for every collection S of filters on $\operatorname{Dst}\mu$ and filter $\mathcal{A}$ on $\operatorname{Src}\mu$, for every funcoid μ.

PROOF.

$$\bigsqcup S \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \bigsqcup S \sqsubseteq \langle\mu\rangle\mathcal{A} \Leftrightarrow \forall \mathcal{F} \in S : \mathcal{F} \sqsubseteq \langle\mu\rangle\mathcal{A} \Leftrightarrow \forall \mathcal{F} \in S : \mathcal{F} \xrightarrow{\mu} \mathcal{A}.$$

□

COROLLARY 1462. $\bigsqcup F \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \forall f \in F : f \xrightarrow{\mu} \mathcal{A}$ for every collection F of funcoids f such that $\operatorname{Dst} f = \operatorname{Dst}\mu$ and filter $\mathcal{A}$ on $\operatorname{Src}\mu$, for every funcoid μ.

PROOF. By corollary 893 we have

$$\bigsqcup F \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \operatorname{im} \bigsqcup F \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \bigsqcup \langle\operatorname{im}\rangle^* F \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \forall f \in \langle\operatorname{im}\rangle^* F : \mathcal{F} \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \forall f \in F : \operatorname{im} f \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \forall f \in F : f \xrightarrow{\mu} \mathcal{A}.$$

□

THEOREM 1463. $f|_{\mathcal{B}_0 \sqcup \mathcal{B}_1} \xrightarrow{\mu} \mathcal{A} \Leftrightarrow f|_{\mathcal{B}_0} \xrightarrow{\mu} \mathcal{A} \wedge f|_{\mathcal{B}_1} \xrightarrow{\mu} \mathcal{A}$. for all filters $\mathcal{A}$, $\mathcal{B}_0$, $\mathcal{B}_1$ and funcoids μ, f and g on suitable sets.

PROOF. As easily follows from distributivity of the lattices of funcoids we have $f|_{\mathcal{B}_0 \sqcup \mathcal{B}_1} = f|_{\mathcal{B}_0} \sqcup f|_{\mathcal{B}_1}$. Thus our theorem follows from the previous corollary. □