$\bigsqcup_{\alpha\in X}]\alpha-\exp\alpha;\alpha+\exp\alpha[\ \sqsupseteq \bigsqcap^{\mathscr{F}}_{X\in\operatorname{up}x}\bigsqcup_{\alpha\in X}]\alpha-\varepsilon;\alpha+\varepsilon[$ for some $\varepsilon>0$ and thus by properties of generalized filter bases ($\left\{\frac{\bigsqcup_{\alpha\in X}]\alpha-\varepsilon;\alpha+\varepsilon[}{X\in\operatorname{up}x}\right\}$ is a filter base) for some $X'\in\operatorname{up}x$

$$\bigsqcup_{\alpha\in X}]\alpha-\exp\alpha;\alpha+\exp\alpha[\ \sqsupseteq \bigsqcup_{\alpha\in X'}]\alpha-\varepsilon;\alpha+\varepsilon[$$

what is impossible by the fact that $\exp\alpha$ goes infinitely small as $\alpha\to-\infty$ and the fact that we can take $X=\mathbb{Z}$ for some x. □

Now prove the general case:

PROOF. Suppose that $K\in\operatorname{up}\bigsqcap^{\mathsf{FCD}}S$ and thus $\langle K\rangle x\sqsupseteq\left\langle\bigsqcap^{\mathsf{FCD}}S\right\rangle x$. We need to prove that there is some $L\in S$ such that $K\sqsupseteq L$.

Take an ultrafilter x.

$\left\langle\bigsqcap^{\mathsf{FCD}}S\right\rangle x=\bigsqcap^{\mathscr{F}}_{L\in S}\langle L\rangle x=\bigsqcap^{\mathscr{F}}_{L\in S,X\in\operatorname{up}x}\langle L\rangle^*X$.

$\langle K\rangle x=\bigsqcap^{\mathscr{F}}_{X\in\operatorname{up}x}\langle K\rangle^*X$.

Then $\langle K\rangle^*X\sqsupseteq\bigsqcap^{\mathscr{F}}_{L\in S,X\in\operatorname{up}x}\langle L\rangle^*X$ for every $X\in\operatorname{up}x$; thus by properties of generalized filter bases ($\left\{\frac{\langle L\rangle^*X}{L\in S}\right\}$ is a filter base);

$\langle K\rangle^*X\sqsupseteq\bigsqcap^{\mathscr{F}}_{X\in\operatorname{up}x}\langle L\rangle^*X$ for some $L\in S$ and thus by properties of generalized filter bases ($\left\{\frac{\langle L\rangle^*X}{X\in\operatorname{up}x}\right\}$ is a filter base) for some $X'\in\operatorname{up}x$

$\langle K\rangle^*X\sqsupseteq\langle L\rangle^*X'\sqsupseteq\langle L\rangle x$.

So $\langle K\rangle x\sqsupseteq\langle L\rangle x$ because this equality holds for every $X\in\operatorname{up}x$. Therefore $K\sqsupseteq L$. □

EXAMPLE 1417. A base of a funcoid which is not a filter base.

PROOF. Consider $f=\mathrm{id}^{\mathsf{FCD}}_{\Omega}$. We know that $\operatorname{up}f$ is not a filter base. But it is a base of a funcoid. □

EXERCISE 1418. Prove that a set S is a filter (on some set) iff

$$\forall X_0,\dots,X_n\in S:\operatorname{up}(X_0\sqcap\dots\sqcap X_n)\subseteq S$$

for every natural n.

A similar statement does *not* hold for funcoids:

EXAMPLE 1419. For a set S of binary relations

$$\forall X_0,\dots,X_n\in S:\operatorname{up}(X_0\sqcap^{\mathsf{FCD}}\dots\sqcap^{\mathsf{FCD}}X_n)\subseteq S$$

does not imply that there exists funcoid f such that $S=\operatorname{up}f$.

PROOF. Take $S_0=\operatorname{up}1^{\mathsf{FCD}}$ (where 1^{FCD} is the identity funcoid on any infinite set) and $S_1=\bigcup_{F\in S_0}\left\{\frac{\operatorname{up}G}{G\in\operatorname{up}^{\Gamma}F}\right\}$ (that is $S_1=\bigcup_{F\in\operatorname{up}^{\Gamma}1^{\mathsf{FCD}}}\operatorname{up}F$).

Both S_0 and S_1 are upper sets. $S_0\neq S_1$ because $1^{\mathsf{FCD}}\in S_0$ and $1^{\mathsf{FCD}}\notin S_1$.

The formula in the example works for $S=S_0$ because $X_0,\dots,X_n\in\operatorname{up}1^{\mathsf{FCD}}$. It also holds for $S=S_1$ by the following reason:

Suppose $X_0,\dots,X_n\in S_1$. Then $X_i\sqsupseteq F_i$ where $F_i\in S_0$. Consequently (take into account that Γ is a sublattice of FCD) $X_0,\dots,X_n\sqsupseteq F_0\sqcap^{\mathsf{FCD}}\dots\sqcap^{\mathsf{FCD}}F_n$ and so $X_0\sqcap^{\mathsf{FCD}}\dots\sqcap^{\mathsf{FCD}}X_n=X_0\sqcap\dots\sqcap X_n\sqsupseteq F_0\sqcap^{\mathsf{FCD}}\dots\sqcap^{\mathsf{FCD}}F_n\sqsupseteq1^{\mathsf{FCD}}$. Thus $X_0\sqcap\dots\sqcap X_n\in\operatorname{up}^{\Gamma}1^{\mathsf{FCD}}\subseteq S_1$; $\operatorname{up}(X_0\sqcap\dots\sqcap X_n)\subseteq S_1$ as S_1 is an upper set.

To finish the proof suppose for the contrary that $\operatorname{up}f_0=S_0$ and $\operatorname{up}f_1=S_1$ for some funcoids f_0 and f_1. In this case $f_0=\bigsqcap^{\mathsf{FCD}}S_0=1^{\mathsf{FCD}}=\bigsqcap^{\mathsf{FCD}}\operatorname{up}^{\Gamma}1^{\mathsf{FCD}}=\bigsqcap^{\mathsf{FCD}}S_1=f_1$ and thus $S_0=S_1$, contradiction. □