16.7. More on properties of funcoids

PROPOSITION 1403. $\Gamma(A,B)$ is the center of lattice $\mathsf{FCD}(A,B)$.

PROOF. Theorem 610. □

PROPOSITION 1404. $\mathrm{up}^{\Gamma(A,B)}(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})$ is defined by the filter base $\left\{\frac{A\times B}{A\in\mathrm{up}\,\mathcal{A},B\in\mathrm{up}\,\mathcal{B}}\right\}$ on the lattice $\Gamma(A,B)$.

PROOF. It follows from the fact that $\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}=\bigsqcap^{\mathsf{FCD}}\left\{\frac{A\times B}{A\in\mathrm{up}\,\mathcal{A},B\in\mathrm{up}\,\mathcal{B}}\right\}$. □

PROPOSITION 1405. $\mathrm{up}^{\Gamma(A,B)}(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})=\mathfrak{F}(\Gamma(A,B))\cap\mathrm{up}(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B})$.

PROOF. It follows from the fact that $\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}=\bigsqcap^{\mathsf{FCD}}\left\{\frac{A\times B}{A\in\mathrm{up}\,\mathcal{A},B\in\mathrm{up}\,\mathcal{B}}\right\}$. □

PROPOSITION 1406. For every $f\in\mathfrak{F}(\Gamma(A,B))$:

1°. $f\circ f$ is defined by the filter base $\left\{\frac{F\circ F}{F\in\mathrm{up}\,f}\right\}$ (if $A=B$);
2°. $f^{-1}\circ f$ is defined by the filter base $\left\{\frac{F^{-1}\circ F}{F\in\mathrm{up}\,f}\right\}$;
3°. $f\circ f^{-1}$ is defined by the filter base $\left\{\frac{F\circ F^{-1}}{F\in\mathrm{up}\,f}\right\}$.

PROOF. I will prove only 1° and 2° because 3° is analogous to 2°.

1°. It's enough to show that $\forall F,G\in\mathrm{up}\,f\exists H\in\mathrm{up}\,f:H\circ H\sqsubseteq G\circ F$. To prove it take $H=F\sqcap G$.

2°. It's enough to show that $\forall F,G\in\mathrm{up}\,f\exists H\in\mathrm{up}\,f:H^{-1}\circ H\sqsubseteq G^{-1}\circ F$. To prove it take $H=F\sqcap G$. Then $H^{-1}\circ H=(F\sqcap G)^{-1}\circ(F\sqcap G)\sqsubseteq G^{-1}\circ F$. □

THEOREM 1407. For every sets A, B, C if $g,h\in\mathfrak{F}\Gamma(A,B)$ then

1°. $f\circ(g\sqcup h)=f\circ g\sqcup f\circ h$;
2°. $(g\sqcup h)\circ f=g\circ f\sqcup h\circ f$.

PROOF. It follows from the order isomorphism above, which preserves composition. □

THEOREM 1408. $f\cap g=f\sqcap^{\mathsf{FCD}}g$ if $f,g\in\Gamma(A,B)$.

PROOF. Let $f=X_0\times Y_0\cup\ldots\cup X_n\times Y_n$ and $g=X'_0\times Y'_0\cup\ldots\cup X'_m\times Y'_m$. Then

$$f\cap g=\bigcup_{i=0,\ldots,n,j=0,\ldots,m}((X_i\times Y_i)\cap(X'_j\times Y'_j))=\bigcup_{i=0,\ldots,n,j=0,\ldots,m}((X_i\cap X'_j)\times(Y_i\cap Y'_j)).$$

But $f=X_0\times Y_0\sqcup^{\mathsf{FCD}}\ldots\sqcup^{\mathsf{FCD}}X_n\times Y_n$ and $g=X'_0\times Y'_0\sqcup^{\mathsf{FCD}}\ldots\sqcup^{\mathsf{FCD}}X'_m\times Y'_m$;

$$f\sqcap^{\mathsf{FCD}}g=\bigsqcup_{i=0,\ldots,n,j=0,\ldots,m}((X_i\times Y_i)\sqcap^{\mathsf{FCD}}(X'_j\times Y'_j))=\bigsqcup_{i=0,\ldots,n,j=0,\ldots,m}((X_i\sqcap X'_j)\times^{\mathsf{FCD}}(Y_i\sqcap Y'_j)).$$

COROLLARY 1409. If X and Y are finite binary relations, then

1°. $X\sqcap^{\mathsf{FCD}}Y=X\sqcap Y$;
2°. $(\top\setminus X)\sqcap^{\mathsf{FCD}}(\top\setminus Y)=(\top\setminus X)\sqcap(\top\setminus Y)$;
3°. $X\sqcap^{\mathsf{FCD}}(\top\setminus Y)=X\sqcap(\top\setminus Y)$.