Proof. For every filter $\mathcal{X} \in \mathscr{F}(\operatorname{Src} f)$ we have $\langle(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f\rangle\mathcal{X} = \bigsqcap^{\mathscr{F}}_{F\in\mathrm{up}(\mathsf{RLD})_{\mathrm{in}} f}\langle F\rangle\mathcal{X} = \bigsqcap^{\mathscr{F}}_{F\in\mathrm{up}^{\Gamma(\operatorname{Src} f,\operatorname{Dst} f)} f}\langle F\rangle\mathcal{X}$.

Obviously $\bigsqcap^{\mathscr{F}}_{F\in\mathrm{up}^{\Gamma(\operatorname{Src} f,\operatorname{Dst} f)} f}\langle F\rangle\mathcal{X} \sqsupseteq \langle f\rangle\mathcal{X}$. So $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f \sqsupseteq f$.

Let $Y \in \mathrm{up}\langle f\rangle\mathcal{X}$. Then (proposition above) there exists $A \in \mathrm{up}\,\mathcal{X}$ such that $Y \in \mathrm{up}\langle f\rangle A$.

Thus $A\times Y \sqcup \overline{A}\times\top \in \mathrm{up}\, f$. So $\langle(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f\rangle\mathcal{X} = \bigsqcap^{\mathscr{F}}_{F\in\mathrm{up}^{\Gamma(\operatorname{Src} f,\operatorname{Dst} f)} f}\langle F\rangle\mathcal{X} \sqsubseteq \langle A\times Y \sqcup \overline{A}\times\top\rangle\mathcal{X} = Y$. So $Y \in \mathrm{up}\langle(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f\rangle\mathcal{X}$ that is $\langle f\rangle\mathcal{X} \sqsupseteq \langle(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f\rangle\mathcal{X}$ that is $f \sqsupseteq (\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f$. □

## 16.6. Some additional properties

Proposition 1397. For every funcoid $f \in \mathsf{FCD}(A, B)$ (for sets $A$, $B$):

1°. $\operatorname{dom} f = \bigsqcap^{\mathscr{F}(A)}\langle\operatorname{dom}\rangle^{*}\,\mathrm{up}^{\Gamma(A,B)} f$;
2°. $\operatorname{im} f = \bigsqcap^{\mathscr{F}(B)}\langle\operatorname{im}\rangle^{*}\,\mathrm{up}^{\Gamma(A,B)} f$.

Proof. Take $\left\{\frac{X\times Y}{X\in\mathscr{P}A, Y\in\mathscr{P}B, X\times Y\sqsupseteq f}\right\} \subseteq \mathrm{up}^{\Gamma(A,B)} f$. I leave the rest reasoning as an exercise. □

Theorem 1398. For every reloid $f$ and $\mathcal{X} \in \mathscr{F}(\operatorname{Src} f)$, $\mathcal{Y} \in \mathscr{F}(\operatorname{Dst} f)$:

1°. $\mathcal{X}\ [(\mathsf{FCD})f]\ \mathcal{Y} \Leftrightarrow \forall F \in \mathrm{up}^{\Gamma(\operatorname{Src} f,\operatorname{Dst} f)} f : \mathcal{X}\ [F]\ \mathcal{Y}$;
2°. $\langle(\mathsf{FCD})f\rangle\mathcal{X} = \bigsqcap^{\mathscr{F}}_{F\in\mathrm{up}^{\Gamma(\operatorname{Src} f,\operatorname{Dst} f)} f}\langle F\rangle\mathcal{X}$.

Proof.

1°.

$$\begin{aligned}&\forall F \in \mathrm{up}^{\Gamma(\operatorname{Src} f,\operatorname{Dst} f)} f : \mathcal{X}\ [F]\ \mathcal{Y} \Leftrightarrow\\ &\qquad\forall F \in \mathrm{up}^{\Gamma(\operatorname{Src} f,\operatorname{Dst} f)} f : (\mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y}) \sqcap F \neq \bot \Leftrightarrow (*)\\ &\qquad(\mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y}) \sqcap \bigsqcap^{\mathsf{FCD}} \mathrm{up}^{\Gamma(\operatorname{Src} f,\operatorname{Dst} f)} f \neq \bot \Leftrightarrow\\ &\qquad\mathcal{X}\left[\bigsqcap^{\mathsf{FCD}} \mathrm{up}^{\Gamma(\operatorname{Src} f,\operatorname{Dst} f)} f\right]\mathcal{Y} \Leftrightarrow \mathcal{X}\ [(\mathsf{FCD})f]\ \mathcal{Y}.\end{aligned}$$

(*) by properties of generalized filter bases, taking into account that funcoids are isomorphic to filters.

2°. $\bigsqcap^{\mathscr{F}}_{F\in\mathrm{up}^{\Gamma(\operatorname{Src} f,\operatorname{Dst} f)} f}\langle F\rangle a = \left\langle\bigsqcap^{\mathsf{FCD}} \mathrm{up}^{\Gamma(\operatorname{Src} f,\operatorname{Dst} f)} f\right\rangle a = \langle(\mathsf{FCD})f\rangle a$ for every ultrafilter $a$.

It remains to prove that the function

$$\varphi = \lambda\mathcal{X} \in \mathscr{F}(\operatorname{Src} f) : \bigsqcap^{\mathscr{F}}_{F\in\mathrm{up}^{\Gamma(\operatorname{Src} f,\operatorname{Dst} f)} f}\langle F\rangle\mathcal{X}$$

is a component of a funcoid (from what follows that $\varphi = \langle(\mathsf{FCD})f\rangle$). To prove this, it's enough to show that it preserves finite joins and filtered meets.