16.4. Associativity over composition

LEMMA 1387. $\prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(A,C)}(g \circ f) = \left(\prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(B,C)} g\right) \circ \left(\prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(B,C)}\right)$ for every $f \in \mathfrak{F}(\Gamma(A,B))$, $g \in \mathfrak{F}(\Gamma(B,C))$ (for every sets A, B, C).

PROOF. If $K \in \operatorname{up} \prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(A,C)}(g \circ f)$ then $K \supseteq G \circ F$ for some $F \in f$, $G \in g$. But $F \in \operatorname{up}^{\Gamma(A,B)} f$, thus

$$F \in \prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(A,B)} f$$

and similarly

$$G \in \prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(B,C)} g.$$

So we have

$$K \supseteq G \circ F \in \operatorname{up}\left(\left(\prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(B,C)} g\right) \circ \left(\prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(A,B)} f\right)\right).$$

Let now

$$K \in \operatorname{up}\left(\left(\prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(B,C)} g\right) \circ \left(\prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(A,B)} f\right)\right).$$

Then there exist $F \in \operatorname{up} \prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(A,B)} f$ and $G \in \operatorname{up} \prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(B,C)} g$ such that $K \supseteq G \circ F$. By properties of generalized filter bases we can take $F \in \operatorname{up}^{\Gamma(A,B)} f$ and $G \in \operatorname{up}^{\Gamma(B,C)} g$. Thus $K \in \operatorname{up}^{\Gamma(A,C)}(g \circ f)$ and so $K \in \operatorname{up} \prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(A,C)}(g \circ f)$. □

LEMMA 1388. $(\mathsf{RLD})_{\mathrm{in}} X = X$ for $X \in \Gamma(A,B)$.

PROOF. $X = X_0 \times Y_0 \cup \ldots \cup X_n \times Y_n = (X_0 \times^{\mathsf{FCD}} Y_0) \sqcup^{\mathsf{FCD}} \ldots \sqcup^{\mathsf{FCD}} (X_n \times^{\mathsf{FCD}} Y_n)$.

$$\begin{aligned}(\mathsf{RLD})_{\mathrm{in}} X = {} & \\ & (\mathsf{RLD})_{\mathrm{in}}(X_0 \times^{\mathsf{FCD}} Y_0) \sqcup^{\mathsf{RLD}} \ldots \sqcup^{\mathsf{RLD}} (\mathsf{RLD})_{\mathrm{in}}(X_n \times^{\mathsf{FCD}} Y) = \\ & (X_0 \times^{\mathsf{RLD}} Y_0) \sqcup^{\mathsf{RLD}} \ldots \sqcup^{\mathsf{RLD}} (X_n \times^{\mathsf{RLD}} Y_n) = \\ & X_0 \times Y_0 \cup \ldots \cup X_n \times Y_n = X.\end{aligned}$$

□

LEMMA 1389. $\prod^{\mathsf{RLD}} f = (\mathsf{RLD})_{\mathrm{in}} \prod^{\mathsf{FCD}} f$ for every filter $f \in \mathfrak{F}\Gamma(A,B)$.

PROOF.

$$(\mathsf{RLD})_{\mathrm{in}} \prod^{\mathsf{FCD}} f = \prod^{\mathsf{RLD}} \langle (\mathsf{RLD})_{\mathrm{in}} \rangle^* f = (\text{by the previous lemma}) = \prod^{\mathsf{RLD}} f.$$

□

LEMMA 1390.

1°. $f \mapsto \prod^{\mathsf{RLD}} \operatorname{up} f$ and $\mathcal{A} \mapsto \Gamma(A,B) \cap \operatorname{up} \mathcal{A}$ are mutually inverse bijections between $\mathfrak{F}\Gamma(A,B)$ and a subset of reloids.

2°. These bijections preserve composition.

PROOF.

1°. That they are mutually inverse bijections is obvious.