PROOF. Let $a \in X$. Then

$$[a] = \left\{ \frac{b \in \bigcup Q}{\forall X' \in Q : (a \in X' \Leftrightarrow b \in X')} \right\} \subseteq \left\{ \frac{b \in \bigcup Q}{a \in X \Leftrightarrow b \in X} \right\} = \left\{ \frac{b \in \bigcup Q}{b \in X} \right\} = X.$$

But $[a] \in \mathfrak{R}(Q)$.

$X \cap Y \neq \emptyset$ follows from $Y \subseteq X$ by the previous proposition. □

PROPOSITION 1377. If $X \in Q$ then $X = \bigcup(\mathfrak{R}(Q) \cap \mathscr{P}X)$.

PROOF. $\bigcup(\mathfrak{R}(Q) \cap \mathscr{P}X) \subseteq X$ is obvious.

Let $x \in X$. Then there is $Y \in \mathfrak{R}(Q)$ such that $x \in Y$. We have $Y \subseteq X$ that is $Y \in \mathscr{P}X$ by a proposition above. So $x \in Y$ where $Y \in \mathfrak{R}(Q) \cap \mathscr{P}X$ and thus $x \in \bigcup(\mathfrak{R}(Q) \cap \mathscr{P}X)$. We have $X \subseteq \bigcup(\mathfrak{R}(Q) \cap \mathscr{P}X)$. □

16.2. Finite unions of Cartesian products

Let A, B be sets.

I will denote $\overline{X} = A \setminus X$.

Let denote $\Gamma(A, B)$ the set of all finite unions $X_0 \times Y_0 \cup \ldots \cup X_{n-1} \times Y_{n-1}$ of Cartesian products, where $n \in \mathbb{N}$ and $X_i \in \mathscr{P}A$, $Y_i \in \mathscr{P}B$ for every $i = 0, \ldots, n-1$.

PROPOSITION 1378. The following sets are pairwise equal:

1°. $\Gamma(A, B)$;
2°. the set of all sets of the form $\bigcup_{X \in S}(X \times Y_X)$ where S are finite collections on A and $Y_X \in \mathscr{P}B$ for every $X \in S$;
3°. the set of all sets of the form $\bigcup_{X \in S}(X \times Y_X)$ where S are finite partitions of A and $Y_X \in \mathscr{P}B$ for every $X \in S$;
4°. the set of all finite unions $\bigcup_{(X,Y) \in \sigma}(X \times Y)$ where σ is a relation between a partition of A and a partition of B (that is $\operatorname{dom}\sigma$ is a partition of A and $\operatorname{im}\sigma$ is a partition of B).
5°. the set of all finite intersections $\bigcap_{i=0,\ldots,n-1}\left(X_i \times Y_i \cup \overline{X_i} \times B\right)$ where $n \in \mathbb{N}$ and $X_i \in \mathscr{P}A$, $Y_i \in \mathscr{P}B$ for every $i = 0, \ldots, n-1$.

PROOF.

1°⊇2°, 2°⊇3°. Obvious.

1°⊆2°. Let $Q \in \Gamma(A, B)$. Then $Q = X_0 \times Y_0 \cup \ldots \cup X_{n-1} \times Y_{n-1}$. Denote $S = \{X_0, \ldots, X_{n-1}\}$. We have $Q = \bigcup_{X' \in S}\left(X' \times \bigcup_{i=0,\ldots,n-1}\left\{\frac{Y_i}{X_i = X'}\right\}\right) \in$ 2°.

2°⊆3°. Let $Q = \bigcup_{X \in S}(X \times Y_X)$ where S is a finite collection on A and $Y_X \in \mathscr{P}B$ for every $X \in S$. Let

$$P = \bigcup_{X' \in \mathfrak{R}(S)}\left(X' \times \bigcup_{X \in S}\left\{\frac{Y_X}{\exists X \in S : X' \subseteq X}\right\}\right).$$

To finish the proof let's show $P = Q$.

$\langle P\rangle^*\{x\} = \bigcup_{X \in S}\left\{\frac{Y_X}{\exists X \in S : X' \subseteq X}\right\}$ where $x \in X'$.

Thus $\langle P\rangle^*\{x\} = \bigcup\left\{\frac{Y_X}{\exists X \in S : x \in X}\right\} = \langle Q\rangle^*\{x\}$. So $P = Q$.

4°⊆3°. $\bigcup_{(X,Y) \in \sigma}(X \times Y) = \bigcup_{X \in \operatorname{dom}\sigma}\left(X \times \bigcup\left\{\frac{Y \in \mathscr{P}B}{(X,Y) \in \sigma}\right\}\right) \in$ 3°.