CHAPTER 16

Funcoids are filters

The motto of this chapter is: "Funcoids are filters on a (boolean) lattice."

16.1. Rearrangement of collections of sets

Let Q be a set of sets.

Let $\equiv$ be the relation on $\bigcup Q$ defined by the formula

$$a \equiv b \Leftrightarrow \forall X \in Q : (a \in X \Leftrightarrow b \in X).$$

PROPOSITION 1369. $\equiv$ is an equivalence relation on $\bigcup Q$.

PROOF.

Reflexivity. Obvious.

Symmetry. Obvious.

Transitivity. Let $a \equiv b \wedge b \equiv c$. Then $a \in X \Leftrightarrow b \in X \Leftrightarrow c \in X$ for every $X \in Q$. Thus $a \equiv c$.

□

DEFINITION 1370. *Rearrangement* $\mathfrak{R}(Q)$ of Q is the set of equivalence classes of $\bigcup Q$ for $\equiv$.

OBVIOUS 1371. $\bigcup \mathfrak{R}(Q) = \bigcup Q$.

OBVIOUS 1372. $\emptyset \notin \mathfrak{R}(Q)$.

LEMMA 1373. $\operatorname{card} \mathfrak{R}(Q) \leq 2^{\operatorname{card} Q}$.

PROOF. Having an equivalence class C, we can find the set $f \in \mathscr{P}Q$ of all $X \in Q$ such that $a \in X$, for every $a \in C$.

$$b \equiv a \Leftrightarrow \forall X \in Q : (a \in X \Leftrightarrow b \in X) \Leftrightarrow \forall X \in Q : (X \in f \Leftrightarrow b \in X).$$

So $C = \left\{ \frac{b \in \bigcup Q}{b \equiv a} \right\}$ can be restored knowing f. Consequently there are no more than $\operatorname{card} \mathscr{P}Q = 2^{\operatorname{card} Q}$ classes. □

COROLLARY 1374. If Q is finite, then $\mathfrak{R}(Q)$ is finite.

PROPOSITION 1375. If $X \in Q$, $Y \in \mathfrak{R}(Q)$ then $X \cap Y \neq \emptyset \Leftrightarrow Y \subseteq X$.

PROOF. Let $X \cap Y \neq \emptyset$ and $x \in X \cap Y$. Then

$$y \in Y \Leftrightarrow x \equiv y \Leftrightarrow \forall X' \in Q : (x \in X' \Leftrightarrow y \in X') \Rightarrow (x \in X \Leftrightarrow y \in X) \Leftrightarrow y \in X$$

for every y. Thus $Y \subseteq X$.

$Y \subseteq X \Rightarrow X \cap Y \neq \emptyset$ because $Y \neq \emptyset$. □

PROPOSITION 1376. If $\emptyset \neq X \in Q$ then there exists $Y \in \mathfrak{R}(Q)$ such that $Y \subseteq X \wedge X \cap Y \neq \emptyset$.