□

LEMMA 1279. Let $\operatorname{card} X = \operatorname{card} Y$, u be an ultrafilter on X and v be an ultrafilter on Y; let $A \in u$ and $B \in v$. Let $u \div A$ and $v \div B$ be directly isomorphic. Then if $\operatorname{card}(X \setminus A) = \operatorname{card}(Y \setminus B)$ we have u and v directly isomorphic.

PROOF. Arbitrary extend the bijection witnessing being directly isomorphic to the sets $X \setminus A$ and $X \setminus B$. □

THEOREM 1280. If $\operatorname{card} X = \operatorname{card} Y$ then being isomorphic and being directly isomorphic are the same for ultrafilters u on X and v on Y.

PROOF. That if two filters are isomorphic then they are directly isomorphic is obvious.

Let ultrafilters u and v be isomorphic that is there is a bijection $f : A \to B$ where $A \in u$, $B \in v$ witnessing isomorphism of u and v.

If one of the filters u or v is a trivial ultrafilter then the other is also a trivial ultrafilter and as it is easy to show they are directly isomorphic. So we can assume u and v are not trivial ultrafilters.

If $\operatorname{card}(X \setminus A) = \operatorname{card}(Y \setminus B)$ our statement follows from the last lemma.

Now assume without loss of generality $\operatorname{card}(X \setminus A) < \operatorname{card}(Y \setminus B)$.

$\operatorname{card} B = \operatorname{card} Y$ because otherwise $\operatorname{card}(X \setminus A) = \operatorname{card}(Y \setminus B)$.

It is easy to show that there exists $B' \supset B$ such that $\operatorname{card}(X \setminus A) = \operatorname{card}(Y \setminus B')$ and $\operatorname{card} B' = \operatorname{card} B$.

We will find a bijection g from B to B' which witnesses direct isomorphism of v to v itself. Then the composition $g \circ f$ witnesses a direct isomorphism of $u \div A$ and $v \div B'$ and by the lemma u and v are directly isomorphic.

Let $D = B' \setminus B$. We have $D \notin v$.

There exists a set $E \subseteq B$ such that $\operatorname{card} E \geq \operatorname{card} D$ and $E \notin v$.

We have $\operatorname{card} E = \operatorname{card}(D \cup E)$ and thus there exists a bijection $h : E \to D \cup E$.

Let

$$g(x) = \begin{cases} x & \text{if } x \in B \setminus E; \\ h(x) & \text{if } x \in E. \end{cases}$$

$g|_{B \setminus E}$ and $g|_E$ are bijections.

$\operatorname{im}(g|_{B \setminus E}) = B \setminus E$; $\operatorname{im}(g|_E) = \operatorname{im} h = D \cup E$;

$$(D \cup E) \cap (B \setminus E) = (D \cap (B \setminus E)) \cup (E \cap (B \setminus E)) = \emptyset \cup \emptyset = \emptyset.$$

Thus g is a bijection from B to $(B \setminus E) \cup (D \cup E) = B \cup D = B'$.

To finish the proof it's enough to show that $\langle g \rangle^* v = v$. Indeed it follows from $B \setminus E \in v$. □

PROPOSITION 1281.

1°. For every $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have $\mathcal{A} \geq_2 \mathcal{B}$ iff $\mathcal{A} \div A \geq_2 \mathcal{B} \div B$.
2°. For every $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have $\mathcal{A} \geq_1 \mathcal{B}$ iff $\mathcal{A} \div A \geq_1 \mathcal{B} \div B$.

PROOF.

1°. $\mathcal{A} \geq_2 \mathcal{B}$ iff there exist a bijective **Set**-morphism f such that $\mathcal{B} = \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A}$. The equality is obviously preserved replacing $\mathcal{A}$ with $\mathcal{A} \div A$ and $\mathcal{B}$ with $\mathcal{B} \div B$.

2°. $\mathcal{A} \geq_1 \mathcal{B}$ iff there exist a bijective **Set**-morphism f such that $\mathcal{B} \subseteq \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A}$. The equality is obviously preserved replacing $\mathcal{A}$ with $\mathcal{A} \div A$ and $\mathcal{B}$ with $\mathcal{B} \div B$.

□