Thus $g \circ f$ is a morphism of $\mathbf{GreFunc}_1$. Associativity law is evident. $\mathrm{id}_{\mathrm{Base}(\mathcal{A})}$ is the identity morphism of $\mathbf{GreFunc}_1$ for every filter $\mathcal{A}$.

Let $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ be morphisms of $\mathbf{GreFunc}_2$. Then $\mathcal{B} = \left\langle \uparrow^{\mathsf{FCD}} f \right\rangle \mathcal{A}$ and $\mathcal{C} = \left\langle \uparrow^{\mathsf{FCD}} g \right\rangle \mathcal{B}$. So

$$\left\langle \uparrow^{\mathsf{FCD}} (g \circ f) \right\rangle \mathcal{A} = \left\langle \uparrow^{\mathsf{FCD}} g \right\rangle \left\langle \uparrow^{\mathsf{FCD}} f \right\rangle \mathcal{A} = \left\langle \uparrow^{\mathsf{FCD}} g \right\rangle \mathcal{B} = \mathcal{C}.$$

Thus $g \circ f$ is a morphism of $\mathbf{GreFunc}_2$. Associativity law is evident. $\mathrm{id}_{\mathrm{Base}(\mathcal{A})}$ is the identity morphism of $\mathbf{GreFunc}_2$ for every filter $\mathcal{A}$. □

Corollary 1272. $\leq_1$ and $\leq_2$ are preorders.

Theorem 1273. **FuncBij** is a groupoid.

Proof. First let's prove it is a category. Let $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ be morphisms of **FuncBij**. Then $f : \mathrm{Base}(\mathcal{A}) \to \mathrm{Base}(\mathcal{B})$ and $g : \mathrm{Base}(\mathcal{B}) \to \mathrm{Base}(\mathcal{C})$ are bijections and $\mathcal{B} = \left\langle \uparrow^{\mathsf{FCD}} f \right\rangle \mathcal{A}$ and $\mathcal{C} = \left\langle \uparrow^{\mathsf{FCD}} g \right\rangle \mathcal{B}$. Thus $g \circ f : \mathrm{Base}(\mathcal{A}) \to \mathrm{Base}(\mathcal{C})$ is a bijection and $\mathcal{C} = \left\langle \uparrow^{\mathsf{FCD}} (g \circ f) \right\rangle \mathcal{A}$. Thus $g \circ f$ is a morphism of **FuncBij**. $\mathrm{id}_{\mathrm{Base}(\mathcal{A})}$ is the identity morphism of **FuncBij** for every filter $\mathcal{A}$. Thus it is a category.

It remains to prove only that every morphism $f \in \mathrm{Hom}_{\mathbf{FuncBij}}(\mathcal{A}, \mathcal{B})$ has a reverse (for every filters $\mathcal{A}$, $\mathcal{B}$). We have f is a bijection $\mathrm{Base}(\mathcal{A}) \to \mathrm{Base}(\mathcal{B})$ such that for every $C \in \mathscr{P}\,\mathrm{Base}(\mathcal{A})$

$$\langle f \rangle^* C \in \mathcal{B} \Leftrightarrow C \in \mathcal{A}.$$

Then $f^{-1} : \mathrm{Base}(\mathcal{B}) \to \mathrm{Base}(\mathcal{A})$ is a bijection such that for every $C \in \mathscr{P}\,\mathrm{Base}(\mathcal{B})$

$$\left\langle f^{-1} \right\rangle^* C \in \mathcal{A} \Leftrightarrow C \in \mathcal{B}.$$

Thus $f^{-1} \in \mathrm{Hom}_{\mathbf{FuncBij}}(\mathcal{B}, \mathcal{A})$. □

Corollary 1274. Being directly isomorphic is an equivalence relation.

Rudin-Keisler order of ultrafilters is considered in such a book as [40].

Obvious 1275. For the case of ultrafilters being directly isomorphic is the same as being Rudin-Keisler equivalent.

Definition 1276. A filter $\mathcal{A}$ is *isomorphic* to a filter $\mathcal{B}$ iff there exist sets $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $\mathcal{A} \div A$ is directly isomorphic to $\mathcal{B} \div B$.

Obvious 1277. Equivalent filters are isomorphic.

Theorem 1278. Being isomorphic (for small filters) is an equivalence relation.

Proof.

Reflexivity. Because every filter is directly isomorphic to itself.

Symmetry. If filter $\mathcal{A}$ is isomorphic to $\mathcal{B}$ then there exist sets $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $\mathcal{A} \div A$ is directly isomorphic to $\mathcal{B} \div B$ and thus $\mathcal{B} \div B$ is directly isomorphic to $\mathcal{A} \div A$. So $\mathcal{B}$ is isomorphic to $\mathcal{A}$.

Transitivity. Let $\mathcal{A}$ be isomorphic to $\mathcal{B}$ and $\mathcal{B}$ be isomorphic to $\mathcal{C}$. Then exist $A \in \mathcal{A}$, $B_1 \in \mathcal{B}$, $B_2 \in \mathcal{B}$, $C \in \mathcal{C}$ such that there are bijections $f : A \to B_1$ and $g : B_2 \to C$ such that

$$\forall X \in \mathscr{P} A : \left(X \in \mathcal{B} \Leftrightarrow \left\langle f^{-1} \right\rangle^* X \in \mathcal{A}\right) \quad \text{and} \quad \forall X \in \mathscr{P} B_1 : \left(X \in \mathcal{A} \Leftrightarrow \langle f \rangle^* X \in \mathcal{B}\right)$$

and also $\forall X \in \mathscr{P} B_2 : (X \in \mathcal{B} \Leftrightarrow \langle g \rangle^* X \in \mathcal{C})$.

So $g \circ f$ is a bijection from $\left\langle f^{-1} \right\rangle^* (B_1 \cap B_2) \in \mathcal{A}$ to $\langle g \rangle^* (B_1 \cap B_2) \in \mathcal{C}$ such that

$$X \in \mathcal{A} \Leftrightarrow \langle f \rangle^* X \in \mathcal{B} \Leftrightarrow \langle g \rangle^* \langle f \rangle^* X \in \mathcal{C} \Leftrightarrow \langle g \circ f \rangle^* X \in \mathcal{C}.$$

Thus $g \circ f$ establishes a bijection which proves that $\mathcal{A}$ is isomorphic to $\mathcal{C}$.