QUESTION 1251. Under which conditions it's true that join of (α-, β-) totally bounded reloids is also totally bounded?

13.5. Additional predicates

We may consider also the following predicates expressing different kinds of what is intuitively is understood as boundness. Their usefulness is unclear, but I present them for completeness.

- $\operatorname{totBound}_\alpha(f)$
- $\operatorname{totBound}_\beta(f)$
- $\exists n \in \mathbb{N} \forall E \in \operatorname{up} f : \operatorname{thick}_\alpha(E^n)$
- $\exists n \in \mathbb{N} \forall E \in \operatorname{up} f : \operatorname{thick}_\beta(E^n)$
- $\exists n \in \mathbb{N} \forall E \in \operatorname{up} f : \operatorname{thick}_\alpha(E^0 \sqcup \ldots \sqcup E^n)$
- $\exists n \in \mathbb{N} \forall E \in \operatorname{up} f : \operatorname{thick}_\beta(E^0 \sqcup \ldots \sqcup E^n)$
- $\exists n \in \mathbb{N} : \operatorname{totBound}_\alpha(f^n)$
- $\exists n \in \mathbb{N} : \operatorname{totBound}_\beta(f^n)$
- $\exists n \in \mathbb{N} : \operatorname{totBound}_\alpha(f^0 \sqcup \ldots \sqcup f^n)$
- $\exists n \in \mathbb{N} : \operatorname{totBound}_\beta(f^0 \sqcup \ldots \sqcup f^n)$
- $\operatorname{totBound}_\alpha(S(f))$
- $\operatorname{totBound}_\beta(S(f))$

Some of the above defined predicates are equivalent:

PROPOSITION 1252.

- $\exists n \in \mathbb{N} \forall E \in \operatorname{up} f : \operatorname{thick}_\alpha(E^n) \Leftrightarrow \exists n \in \mathbb{N} : \operatorname{totBound}_\alpha(f^n)$.
- $\exists n \in \mathbb{N} \forall E \in \operatorname{up} f : \operatorname{thick}_\beta(E^n) \Leftrightarrow \exists n \in \mathbb{N} : \operatorname{totBound}_\beta(f^n)$.

PROOF. Because for every $E \in \operatorname{up} f$ some $F \in \operatorname{up} f^n$ is a subset of E^n, we have

$$\forall E \in \operatorname{up} f : \operatorname{thick}_\alpha(E^n) \Leftrightarrow \forall F \in \operatorname{up} f^n : \operatorname{thick}_\alpha(F)$$

and likewise for $\operatorname{thick}_\beta$. □

PROPOSITION 1253.

- $\exists n \in \mathbb{N} \forall E \in \operatorname{up} f : \operatorname{thick}_\alpha(E^0 \sqcup \ldots \sqcup E^n) \Leftrightarrow \exists n \in \mathbb{N} : \operatorname{totBound}_\alpha(f^0 \sqcup \ldots \sqcup f^n)$
- $\exists n \in \mathbb{N} \forall E \in \operatorname{up} f : \operatorname{thick}_\beta(E^0 \sqcup \ldots \sqcup E^n) \Leftrightarrow \exists n \in \mathbb{N} : \operatorname{totBound}_\beta(f^0 \sqcup \ldots \sqcup f^n)$

PROOF. It's enough to prove

$$\forall E \in \operatorname{up} f \exists F \in \operatorname{up}(f^0 \sqcup \cdots \sqcup f^n) : F \sqsubseteq E^0 \sqcup \ldots \sqcup E^n \text{ and} \quad (15)$$

$$\forall F \in \operatorname{up}(f^0 \sqcup \cdots \sqcup f^n) \exists E \in \operatorname{up} f : E^0 \sqcup \ldots \sqcup E^n \sqsubseteq F. \quad (16)$$

For the formula (15) take $F = E^0 \sqcup \cdots \sqcup E^n$.

Let's prove (16). Let $F \in \operatorname{up}(f^0 \sqcup \cdots \sqcup f^n)$. Using the fact that $F \in \operatorname{up} f^i$ take $E_i \in \operatorname{up} f$ for $i = 0, \ldots, n$ such that $E_i^i \sqsubseteq F$ (exercise 1004 and properties of generalized filter bases) and then $E = E_0 \sqcap \cdots \sqcap E_n \in \operatorname{up} f$. We have $E^0 \sqcup \ldots \sqcup E^n \sqsubseteq F$. □

PROPOSITION 1254. All predicates in the above list are pairwise equivalent in the case if f is a uniform space.

PROOF. Because $f \circ f = f$ and thus $f^n = f^0 \sqcup \cdots \sqcup f^n = S(f) = f$. □