⇐. For every $\epsilon \in \operatorname{up} f$ we have that $\langle \mathrm{GR}\,\epsilon\rangle^* \{c_0\}, \dots, \langle \mathrm{GR}\,\epsilon\rangle^* \{c_n\}$ covers the space. $\langle \mathrm{GR}\,\epsilon\rangle^* \{c_i\} \times \langle \mathrm{GR}\,\epsilon\rangle^* \{c_i\} \subseteq \mathrm{GR}(\epsilon \circ \epsilon^{-1})$ because for $x \in \langle \mathrm{GR}\,\epsilon\rangle^* \{c_i\}$ (the same as $c_i \in \langle \mathrm{GR}\,\epsilon\rangle^* \{x\}$) we have

$$\left\langle \langle \mathrm{GR}\,\epsilon\rangle^* \{c_i\} \times \langle \mathrm{GR}\,\epsilon\rangle^* \{c_i\}\right\rangle^* \{x\} = \langle \mathrm{GR}\,\epsilon\rangle^* \{c_i\} \subseteq \langle \mathrm{GR}\,\epsilon\rangle^* \left\langle \mathrm{GR}\,\epsilon^{-1}\right\rangle^* \{x\} = \left\langle \mathrm{GR}(\epsilon \circ \epsilon^{-1})\right\rangle^* \{x\}.$$

For every $\epsilon' \in \operatorname{up} f$ exists $\epsilon \in \operatorname{up} f$ such that $\epsilon \circ \epsilon^{-1} \sqsubseteq \epsilon'$ because $f \circ f^{-1} \sqsubseteq f$. Thus for every ϵ' we have $\langle \mathrm{GR}\,\epsilon\rangle^* \{c_i\} \times \langle \mathrm{GR}\,\epsilon\rangle^* \{c_i\} \subseteq \mathrm{GR}\,\epsilon'$ and so $\langle \mathrm{GR}\,\epsilon\rangle^* \{c_0\}, \dots, \langle \mathrm{GR}\,\epsilon\rangle^* \{c_n\}$ is a sought for finite cover.

□

COROLLARY 1247. A uniform space is α-totally bounded iff it is β-totally bounded.

PROOF. From the theorem and the definition of uniform spaces. □

Thus we can say about just *totally bounded* uniform spaces (without specifying whether it is α or β).

13.4. Relationships with other properties

THEOREM 1248. Let μ and ν be endoreloids. Let f be a principal $\mathrm{C}'(\mu, \nu)$ continuous, monovalued, surjective reloid. Then if μ is β-totally bounded then ν is also β-totally bounded.

PROOF. Let φ be the monovalued, surjective function, which induces the reloid f.

We have $\mu \sqsubseteq f^{-1} \circ \nu \circ f$.

Let $F \in \operatorname{up} \nu$. Then there exists $E \in \operatorname{up} \mu$ such that $E \subseteq \varphi^{-1} \circ F \circ \varphi$.

Since μ is β-totally bounded, there exists a finite typed subset A of $\operatorname{Ob} \mu$ such that $\langle \mathrm{GR}\, E\rangle^* A = \operatorname{Ob} \mu$.

We claim $\langle \mathrm{GR}\, F\rangle^* \langle \varphi\rangle^* A = \operatorname{Ob} \nu$.

Indeed let $y \in \operatorname{Ob} \nu$ be an arbitrary point. Since φ is surjective, there exists $x \in \operatorname{Ob} \mu$ such that $\varphi x = y$. Since $\langle \mathrm{GR}\, E\rangle^* A = \operatorname{Ob} \mu$ there exists $a \in A$ such that $a\ (\mathrm{GR}\, E)\ x$ and thus $a\ (\varphi^{-1} \circ F \circ \varphi)\ x$. So $(\varphi a, y) = (\varphi a, \varphi x) \in \mathrm{GR}\, F$. Therefore $y \in \langle \mathrm{GR}\, F\rangle^* \langle \varphi\rangle^* A$. □

THEOREM 1249. Let μ and ν be endoreloids. Let f be a principal $\mathrm{C}''(\mu, \nu)$ continuous, surjective reloid. Then if μ is α-totally bounded then ν is also α-totally bounded.

PROOF. Let φ be the surjective binary relation which induces the reloid f.

We have $f \circ \mu \circ f^{-1} \sqsubseteq \nu$.

Let $F \in \operatorname{up} \nu$. Then there exists $E \in \operatorname{up} \mu$ such that $\varphi \circ E \circ \varphi^{-1} \subseteq F$.

There exists a finite cover S of $\operatorname{Ob} \mu$ such that $\bigcup \left\{ \frac{A \times A}{A \in S} \right\} \subseteq \mathrm{GR}\, E$.

Thus $\varphi \circ \left(\bigcup \left\{ \frac{A \times A}{A \in S} \right\} \right) \circ \varphi^{-1} \subseteq \mathrm{GR}\, F$ that is $\bigcup \left\{ \frac{\langle \varphi\rangle^* A \times \langle \varphi\rangle^* A}{A \in S} \right\} \subseteq \mathrm{GR}\, F$.

It remains to prove that $\left\{ \frac{\langle \varphi\rangle^* A}{A \in S} \right\}$ is a cover of $\operatorname{Ob} \nu$. It is true because φ is a surjection and S is a cover of $\operatorname{Ob} \mu$. □

A stronger statement (principality requirement removed):

CONJECTURE 1250. The image of a uniformly continuous entirely defined monovalued surjective reloid from a (α-, β-)totally bounded endoreloid is also (α-, β-)totally bounded.

Can we remove the requirement to be entirely defined from the above conjecture?