PROOF.

$$S^*(S^*(f)) = \prod^{\mathsf{RLD}}_{R\in\operatorname{up}S^*(f)} S(R) \sqsubseteq \prod^{\mathsf{RLD}}_{R\in\left\{\frac{S(F)}{F\in\operatorname{up}f}\right\}} S(R) = \prod^{\mathsf{RLD}}_{R\in\operatorname{up}f} S(S(R)) = \prod^{\mathsf{RLD}}_{R\in\operatorname{up}f} S(R) = S^*(f).$$

So $S^*(S^*(f)) \sqsubseteq S^*(f)$. That $S^*(S^*(f)) \sqsupseteq S^*(f)$ is obvious. □

COROLLARY 1221. $S^*(S(f)) = S(S^*(f)) = S^*(f)$ for every endoreloid f.

PROOF. Obviously $S^*(S(f)) \sqsupseteq S^*(f)$ and $S(S^*(f)) \sqsupseteq S^*(f)$.
But $S^*(S(f)) \sqsubseteq S^*(S^*(f)) = S^*(f)$ and $S(S^*(f)) \sqsubseteq S^*(S^*(f)) = S^*(f)$. □

CONJECTURE 1222. $S(S(f)) = S(f)$ for

1°. every endoreloid f;
2°. every endofuncoid f.

CONJECTURE 1223. $S(f) \circ S(f) = S(f)$ for every endoreloid f.

THEOREM 1224. $S^*(f) \circ S^*(f) = S(f) \circ S^*(f) = S^*(f) \circ S(f) = S^*(f)$ for every endoreloid f.

PROOF. [2]

It is enough to prove $S^*(f) \circ S^*(f) = S^*(f)$ because $S^*(f) \sqsubseteq S(f) \circ S^*(f) \sqsubseteq S^*(f) \circ S^*(f)$ and likewise for $S^*(f) \circ S(f)$.

$S^*(\mu) \circ S^*(\mu) = \prod^{\mathsf{RLD}}_{F\in\operatorname{up}S^*(\mu)}(F \circ F) =$ (see below) $= \prod^{\mathsf{RLD}}_{X\in\operatorname{up}\mu}(S(X) \circ S(X)) = \prod^{\mathsf{RLD}}_{X\in\operatorname{up}\mu} S(X) = S^*(\mu)$.

$F \in \operatorname{up} S^*(\mu) \Leftrightarrow F \in \operatorname{up} \prod^{\mathscr{F}}_{F\in\operatorname{up}\mu} S(F) \Rightarrow$ (by properties of filter bases) $\Rightarrow \exists X \in \operatorname{up}\mu : F \sqsupseteq S(X) \Rightarrow \exists X \in \operatorname{up}\mu : F \circ F \sqsupseteq S(X) \circ S(X)$ thus

$$\prod^{\mathsf{RLD}}_{F\in\operatorname{up}S^*(\mu)} F \circ F \sqsupseteq \prod^{\mathsf{RLD}}_{X\in\operatorname{up}\mu} (S(X) \circ S(X));$$

$X \in \operatorname{up}\mu \Rightarrow S(X) \in \operatorname{up} S^*(\mu) \Rightarrow \exists F \in \operatorname{up} S^*(\mu) : S(X) \circ S(X) \sqsupseteq F \circ F$ thus

$$\prod^{\mathsf{RLD}}_{F\in\operatorname{up}S^*(\mu)} F \circ F \sqsubseteq \prod^{\mathsf{RLD}}_{X\in\operatorname{up}\mu} (S(X) \circ S(X)).$$

□

CONJECTURE 1225. $S(f) \circ S(f) = S(f)$ for every endofuncoid f.

[2] Can be more succintly proved considering $\mu \mapsto S^*(\mu)$ as a pointfree funcoid?