

- 2°.  $S_1(\mu \cap (A \times A)) \supseteq A \times A$ .  
 3°.  $S_1(\mu \cap (A \times A)) = A \times A$ .  
 4°.  $A$  is connected regarding  $\mu$ .

PROOF.

- 1° $\Rightarrow$ 2°. Let for every  $a, b \in A$  there is a nonzero-length path between  $a$  and  $b$  in  $A$  through  $\mu$ . Then  $a (S_1(\mu \cap (A \times A))) b$  for every  $a, b \in A$ . It is possible only when  $S_1(\mu \cap (A \times A)) \supseteq A \times A$ .  
 3° $\Rightarrow$ 1°. For every two vertices  $a$  and  $b$  we have  $a (S_1(\mu \cap (A \times A))) b$ . So (by the previous) for every two vertices  $a$  and  $b$  there exists a nonzero-length path from  $a$  to  $b$ .  
 3° $\Rightarrow$ 4°. Suppose  $\neg(X [\mu \cap (A \times A)]^* Y)$  for some  $X, Y \in \mathcal{P}\mathcal{U} \setminus \{\emptyset\}$  such that  $X \cup Y = A$ . Then by a lemma  $\neg(X [(\mu \cap (A \times A))^n]^* Y)$  for every  $n \in \mathbb{Z}_+$ . Consequently  $\neg(X [S_1(\mu \cap (A \times A))]^* Y)$ . So  $S_1(\mu \cap (A \times A)) \neq A \times A$ .  
 4° $\Rightarrow$ 3°. If  $\langle S_1(\mu \cap (A \times A)) \rangle^* \{v\} = A$  for every vertex  $v$  then  $S_1(\mu \cap (A \times A)) = A \times A$ . Consider the remaining case when  $V \stackrel{\text{def}}{=} \langle S_1(\mu \cap (A \times A)) \rangle^* \{v\} \subset A$  for some vertex  $v$ . Let  $W = A \setminus V$ . If  $\text{card } A = 1$  then  $S_1(\mu \cap (A \times A)) \supseteq \text{id}_A = A \times A$ ; otherwise  $W \neq \emptyset$ . Then  $V \cup W = A$  and so  $V [\mu]^* W$  what is equivalent to  $V [\mu \cap (A \times A)]^* W$  that is  $\langle \mu \cap (A \times A) \rangle^* V \cap W \neq \emptyset$ . This is impossible because

$$\begin{aligned} \langle \mu \cap (A \times A) \rangle^* V &= \langle \mu \cap (A \times A) \rangle^* \langle S_1(\mu \cap (A \times A)) \rangle^* V = \\ &= \langle (\mu \cap (A \times A))^2 \cup (\mu \cap (A \times A))^3 \cup \dots \cup \rangle^* V \subseteq \langle S_1(\mu \cap (A \times A)) \rangle^* V = V. \end{aligned}$$

- 2° $\Rightarrow$ 3°. Because  $S_1(\mu \cap (A \times A)) \subseteq A \times A$ . □

COROLLARY 1199. A set  $A$  is connected regarding a binary relation  $\mu$  iff it is connected regarding  $\mu \cap (A \times A)$ .

DEFINITION 1200. A *connected component* of a set  $A$  regarding a binary relation  $F$  is a maximal connected subset of  $A$ .

THEOREM 1201. The set  $A$  is partitioned into connected components (regarding every binary relation  $F$ ).

PROOF. Consider the binary relation  $a \sim b \Leftrightarrow a (S(F)) b \wedge b (S(F)) a$ .  $\sim$  is a symmetric, reflexive, and transitive relation. So all points of  $A$  are partitioned into a collection of sets  $Q$ . Obviously each component is (strongly) connected. If a set  $R \subseteq A$  is greater than one of that connected components  $A$  then it contains a point  $b \in B$  where  $B$  is some other connected component. Consequently  $R$  is disconnected. □

PROPOSITION 1202. A set is connected (regarding a binary relation) iff it has one connected component.

PROOF. Direct implication is obvious. Reverse is proved by contradiction. □

### 12.4. Connectedness regarding funcoids and reloids

DEFINITION 1203. *Connectivity reloid*  $S_1^*(\mu) = \prod_{M \in \text{up } \mu}^{\text{RLD}} S_1(M)$  for an endoreloid  $\mu$ .

DEFINITION 1204.  $S^*(\mu)$  for an endoreloid  $\mu$  is defined as follows:

$$S^*(\mu) = \prod_{M \in \text{up } \mu}^{\text{RLD}} S(M).$$