THEOREM 1175. If f is a monovalued and entirely defined morphism of a partially ordered dagger precategory then

$$f \in \mathrm{C}'(\mu,\nu) \Leftrightarrow f \in \mathrm{C}(\mu,\nu) \Leftrightarrow f \in \mathrm{C}''(\mu,\nu).$$

PROOF. From two previous propositions. □

The classical general topology theorem that uniformly continuous function from a uniform space to an other uniform space is proximity-continuous regarding the proximities generated by the uniformities, generalized for reloids and funcoids takes the following form:

THEOREM 1176. If an entirely defined morphism of the category of reloids $f \in \mathrm{C}''(\mu,\nu)$ for some endomorphisms μ and ν of the category of reloids, then $(\mathsf{FCD})f \in \mathrm{C}'((\mathsf{FCD})\mu, (\mathsf{FCD})\nu)$.

EXERCISE 1177. I leave a simple exercise for the reader to prove the last theorem.

THEOREM 1178. Let μ and ν be endomorphisms of some partially ordered dagger precategory and $f \in \mathrm{Hom}(\mathrm{Ob}\,\mu, \mathrm{Ob}\,\nu)$ be a monovalued, entirely defined morphism. Then

$$f \in \mathrm{C}(\mu,\nu) \Leftrightarrow f \in \mathrm{C}(\mu^\dagger,\nu^\dagger).$$

PROOF. $f \circ \mu \sqsubseteq \nu \circ f \Leftrightarrow \mu \sqsubseteq f^\dagger \circ \nu \circ f \Rightarrow \mu \circ f^\dagger \sqsubseteq f^\dagger \circ \nu \circ f \circ f^\dagger \Rightarrow \mu \circ f^\dagger \sqsubseteq f^\dagger \circ \nu \Leftrightarrow f \circ \mu^\dagger \sqsubseteq \nu^\dagger \circ f \Rightarrow f^\dagger \circ f \circ \mu^\dagger \sqsubseteq f^\dagger \circ \nu^\dagger \circ f \Rightarrow \mu^\dagger \sqsubseteq f^\dagger \circ \nu^\dagger \circ f \Leftrightarrow \mu \sqsubseteq f^\dagger \circ \nu \circ f$.

Thus $f \circ \mu \sqsubseteq \nu \circ f \Leftrightarrow \mu \Leftrightarrow f \circ \mu^\dagger \sqsubseteq \nu^\dagger \circ f$. □

11.3. Continuity for topological spaces

PROPOSITION 1179. The following are pairwise equivalent for funcoids μ, ν and a monovalued, entirely defined morphism $f \in \mathrm{Hom}(\mathrm{Ob}\,\mu, \mathrm{Ob}\,\nu)$:

1°. $\forall A \in \mathscr{T}\,\mathrm{Ob}\,\mu, B \in \mathrm{up}\langle\nu\rangle\langle f\rangle^* A : \langle f^{-1}\rangle^* B \in \mathrm{up}\langle\mu\rangle^* A$.
2°. $f \in \mathrm{C}(\mu,\nu)$.
3°. $f \in \mathrm{C}(\mu^{-1},\nu^{-1})$.

PROOF.

2°⇔3°. By general $f \circ \mu \sqsubseteq \nu \circ f \Leftrightarrow f \circ \mu^\dagger \sqsubseteq \nu^\dagger \circ f$ formula above.

1°⇔2°. 1° is equivalent to $\left\langle\langle f^{-1}\rangle^*\right\rangle^* \mathrm{up}\langle\nu\rangle\langle f\rangle^* A \subseteq \mathrm{up}\langle\mu\rangle^* A$ equivalent to $\langle\nu\rangle\langle f\rangle^* A \sqsupseteq \langle f\rangle\langle\mu\rangle^* A$ (used "Orderings of filters" chapter).

□

COROLLARY 1180. The following are pairwise equivalent for topological spaces μ, ν and a monovalued, entirely defined morphism $f \in \mathrm{Hom}(\mathrm{Ob}\,\mu, \mathrm{Ob}\,\nu)$:

1°. $\forall x \in \mathrm{Ob}\,\mu, B \in \mathrm{up}\langle\nu\rangle\langle f\rangle^*\{x\} : \langle f^{-1}\rangle^* B \in \mathrm{up}\langle\mu\rangle^*\{x\}$.
2°. Preimages (by f) of open sets are open.
3°. $f \in \mathrm{C}(\mu,\nu)$ that is $\langle f\rangle\langle\mu\rangle^*\{x\} \sqsubseteq \langle\nu\rangle\langle f\rangle^*\{x\}$ for every $x \in \mathrm{Ob}\,\mu$.
4°. $f \in \mathrm{C}(\mu^{-1},\nu^{-1})$ that is $\langle f\rangle\langle\mu^{-1}\rangle^* A \sqsubseteq \langle\nu^{-1}\rangle\langle f\rangle^* A$ for every $A \in \mathscr{T}\,\mathrm{Ob}\,\mu$.

PROOF. 2° from the previous proposition is equivalent to $\langle f\rangle\langle\mu\rangle^*\{x\} \sqsubseteq \langle\nu\rangle\langle f\rangle^*\{x\}$ equivalent to $\left\langle\langle f^{-1}\rangle^*\right\rangle^* \mathrm{up}\langle\nu\rangle\langle f\rangle^*\{x\} \subseteq \mathrm{up}\langle\mu\rangle^*\{x\}$ for every $x \in \mathrm{Ob}\,\mu$, equivalent to 1° (used "Orderings of filters" chapter).

It remains to prove 3°⇔2°.

3°⇒2°. Let B be an open set in ν. For every $x \in \langle f^{-1}\rangle^* B$ we have $f(x) \in B$ that is B is a neighborhood of $f(x)$, thus $\langle f^{-1}\rangle^* B$ is a neighborhood of x. We have proved that $\langle f^{-1}\rangle^* B$ is open.