11.2. Our three definitions of continuity

I have expressed different kinds of continuity with simple algebraic formulas hiding the complexity of traditional epsilon-delta notation behind a smart algebra. Let's summarize these three algebraic formulas:

Let μ and ν be endomorphisms of some partially ordered precategory. Continuous functions can be defined as these morphisms f of this precategory which conform to the following formula:

$$f \in \mathrm{C}(\mu, \nu) \Leftrightarrow f \in \operatorname{Hom}(\operatorname{Ob}\mu, \operatorname{Ob}\nu) \wedge f \circ \mu \sqsubseteq \nu \circ f.$$

If the precategory is a partially ordered dagger precategory then continuity also can be defined in two other ways:

$$f \in \mathrm{C}'(\mu, \nu) \Leftrightarrow f \in \operatorname{Hom}(\operatorname{Ob}\mu, \operatorname{Ob}\nu) \wedge \mu \sqsubseteq f^\dagger \circ \nu \circ f;$$
$$f \in \mathrm{C}''(\mu, \nu) \Leftrightarrow f \in \operatorname{Hom}(\operatorname{Ob}\mu, \operatorname{Ob}\nu) \wedge f \circ \mu \circ f^\dagger \sqsubseteq \nu.$$

REMARK 1171. In the examples (above) about funcoids and reloids the "dagger functor" is the reverse of a funcoid or reloid, that is $f^\dagger = f^{-1}$.

PROPOSITION 1172. Every of these three definitions of continuity forms a wide sub-precategory (wide subcategory if the original precategory is a category).

PROOF.

C. Let $f \in \mathrm{C}(\mu, \nu)$, $g \in \mathrm{C}(\nu, \pi)$. Then $f \circ \mu \sqsubseteq \nu \circ f$, $g \circ \nu \sqsubseteq \pi \circ g$, $g \circ f \circ \mu \sqsubseteq g \circ \nu \circ f \sqsubseteq \pi \circ g \circ f$. So $g \circ f \in \mathrm{C}(\mu, \pi)$. $1_{\operatorname{Ob}\mu} \in \mathrm{C}(\mu, \mu)$ is obvious.

C′. Let $f \in \mathrm{C}'(\mu, \nu)$, $g \in \mathrm{C}'(\nu, \pi)$. Then $\mu \sqsubseteq f^\dagger \circ \nu \circ f$, $\nu \sqsubseteq g^\dagger \circ \pi \circ g$;

$$\mu \sqsubseteq f^\dagger \circ g^\dagger \circ \pi \circ g \circ f; \quad \mu \sqsubseteq (g \circ f)^\dagger \circ \pi \circ (g \circ f).$$

So $g \circ f \in \mathrm{C}'(\mu, \pi)$. $1_{\operatorname{Ob}\mu} \in \mathrm{C}'(\mu, \mu)$ is obvious.

C″. Let $f \in \mathrm{C}''(\mu, \nu)$, $g \in \mathrm{C}''(\nu, \pi)$. Then $f \circ \mu \circ f^\dagger \sqsubseteq \nu$, $g \circ \nu \circ g^\dagger \sqsubseteq \pi$;

$$g \circ f \circ \mu \circ f^\dagger \circ g^\dagger \sqsubseteq \pi; \quad (g \circ f) \circ \mu \circ (g \circ f)^\dagger \sqsubseteq \pi.$$

So $g \circ f \in \mathrm{C}''(\mu, \pi)$. $1_{\operatorname{Ob}\mu} \in \mathrm{C}''(\mu, \mu)$ is obvious.

□

PROPOSITION 1173. For a monovalued morphism f of a partially ordered dagger category and its endomorphisms μ and ν

$$f \in \mathrm{C}'(\mu, \nu) \Rightarrow f \in \mathrm{C}(\mu, \nu) \Rightarrow f \in \mathrm{C}''(\mu, \nu).$$

PROOF. Let $f \in \mathrm{C}'(\mu, \nu)$. Then $\mu \sqsubseteq f^\dagger \circ \nu \circ f$;

$$f \circ \mu \sqsubseteq f \circ f^\dagger \circ \nu \circ f \sqsubseteq 1_{\operatorname{Dst} f} \circ \nu \circ f = \nu \circ f; \quad f \in \mathrm{C}(\mu, \nu).$$

Let $f \in \mathrm{C}(\mu, \nu)$. Then $f \circ \mu \sqsubseteq \nu \circ f$;

$$f \circ \mu \circ f^\dagger \sqsubseteq \nu \circ f \circ f^\dagger \sqsubseteq \nu \circ 1_{\operatorname{Dst} f} = \nu; \quad f \in \mathrm{C}''(\mu, \nu).$$

□

PROPOSITION 1174. For an entirely defined morphism f of a partially ordered dagger category and its endomorphisms μ and ν

$$f \in \mathrm{C}''(\mu, \nu) \Rightarrow f \in \mathrm{C}(\mu, \nu) \Rightarrow f \in \mathrm{C}'(\mu, \nu).$$

PROOF. Let $f \in \mathrm{C}''(\mu, \nu)$. Then $f \circ \mu \circ f^\dagger \sqsubseteq \nu$; $f \circ \mu \circ f^\dagger \circ f \sqsubseteq \nu \circ f$; $f \circ \mu \circ 1_{\operatorname{Src} f} \sqsubseteq \nu \circ f$; $f \circ \mu \sqsubseteq \nu \circ f$; $f \in \mathrm{C}(\mu, \nu)$.

Let $f \in \mathrm{C}(\mu, \nu)$. Then $f \circ \mu \sqsubseteq \nu \circ f$; $f^\dagger \circ f \circ \mu \sqsubseteq f^\dagger \circ \nu \circ f$; $1_{\operatorname{Src}\mu} \circ \mu \sqsubseteq f^\dagger \circ \nu \circ f$; $\mu \sqsubseteq f^\dagger \circ \nu \circ f$; $f \in \mathrm{C}'(\mu, \nu)$. □

For entirely defined monovalued morphisms our three definitions of continuity coincide: