CHAPTER 11

Continuous morphisms

This chapter uses the apparatus from the section "Partially ordered dagger categories".

11.1. Traditional definitions of continuity

In this section we will show that having a funcoid or reloid $\uparrow f$ corresponding to a function f we can express continuity of it by the formula $\uparrow f \circ \mu \sqsubseteq \nu \circ \uparrow f$ (or similar formulas) where μ and ν are some spaces.

11.1.1. Pretopology. Let (A, cl_A) and (B, cl_B) be preclosure spaces. Then by definition a function $f : A \to B$ is continuous iff $f \operatorname{cl}_A(X) \subseteq \operatorname{cl}_B(fX)$ for every $X \in \mathscr{P}A$. Let now μ and ν be endofuncoids corresponding correspondingly to cl_A and cl_B. Then the condition for continuity can be rewritten as

$$\uparrow^{\mathsf{FCD}(\operatorname{Ob}\mu, \operatorname{Ob}\nu)} f \circ \mu \sqsubseteq \nu \circ \uparrow^{\mathsf{FCD}(\operatorname{Ob}\mu, \operatorname{Ob}\nu)} f.$$

11.1.2. Proximity spaces. Let μ and ν be proximity spaces (which I consider a special case of endofuncoids). By definition a **Set**-morphism f is a proximity-continuous map from μ to ν iff

$$\forall X, Y \in \mathscr{T}(\operatorname{Ob}\mu) : (X \; [\mu]^* \; Y \Rightarrow \langle f\rangle^* X \; [\nu]^* \; \langle f\rangle^* Y).$$

Equivalently transforming this formula we get

$$\forall X, Y \in \mathscr{T}(\operatorname{Ob}\mu) : (X \; [\mu]^* \; Y \Rightarrow \langle f\rangle \uparrow X \; [\nu] \; \langle f\rangle \uparrow Y);$$
$$\forall X, Y \in \mathscr{T}(\operatorname{Ob}\mu) : (X \; [\mu]^* \; Y \Rightarrow \uparrow X \; \left[f^{-1} \circ \nu \circ f\right] \uparrow Y);$$
$$\forall X, Y \in \mathscr{T}(\operatorname{Ob}\mu) : (X \; [\mu]^* \; Y \Rightarrow X \; \left[f^{-1} \circ \nu \circ f\right]^* \; Y);$$
$$\mu \sqsubseteq f^{-1} \circ \nu \circ f.$$

So a function f is proximity continuous iff $\mu \sqsubseteq f^{-1} \circ \nu \circ f$.

11.1.3. Uniform spaces. Uniform spaces are a special case of endoreloids.

Let μ and ν be uniform spaces. By definition a **Set**-morphism f is a uniformly continuous map from μ to ν iff

$$\forall \varepsilon \in \operatorname{up}\nu \exists \delta \in \operatorname{up}\nu \forall (x, y) \in \delta : (fx, fy) \in \varepsilon.$$

Equivalently transforming this formula we get:

$$\forall \epsilon \in \operatorname{up}\nu \exists \delta \in \operatorname{up}\mu \forall (x, y) \in \delta : \{(fx, fy)\} \subseteq \epsilon;$$
$$\forall \epsilon \in \operatorname{up}\nu \exists \delta \in \operatorname{up}\mu \forall (x, y) \in \delta : f \circ \{(x, y)\} \circ f^{-1} \subseteq \epsilon;$$
$$\forall \epsilon \in \operatorname{up}\nu \exists \delta \in \operatorname{up}\mu : f \circ \delta \circ f^{-1} \subseteq \epsilon;$$
$$\forall \epsilon \in \operatorname{up}\nu : \uparrow^{\mathsf{RLD}(\operatorname{Ob}\mu, \operatorname{Ob}\nu)} f \circ \mu \circ (\uparrow^{\mathsf{RLD}(\operatorname{Ob}\mu, \operatorname{Ob}\nu)} f)^{-1} \sqsubseteq \uparrow^{\mathsf{RLD}(\operatorname{Ob}\mu, \operatorname{Ob}\nu)} \epsilon;$$
$$\uparrow^{\mathsf{RLD}(\operatorname{Ob}\mu, \operatorname{Ob}\nu)} f \circ \mu \circ (\uparrow^{\mathsf{RLD}(\operatorname{Ob}\mu, \operatorname{Ob}\nu)} f)^{-1} \sqsubseteq \nu.$$

So a function f is uniformly continuous iff $f \circ \mu \circ f^{-1} \sqsubseteq \nu$.