9.8. Double filtrators

Below I show that it's possible to describe (FCD), $(\mathsf{RLD})_{\mathrm{out}}$, and $(\mathsf{RLD})_{\mathrm{in}}$ entirely in terms of filtrators (order). This seems not to lead to really interesting results but it's curious.

DEFINITION 1147. *Double filtrator* is a triple $(\mathfrak{A}, \mathfrak{B}, \mathfrak{Z})$ of posets such that $\mathfrak{Z}$ is a sub-poset of both $\mathfrak{A}$ and $\mathfrak{B}$.

In other words, a double filtrator $(\mathfrak{A}, \mathfrak{B}, \mathfrak{Z})$ is a triple such that both $(\mathfrak{A}, \mathfrak{Z})$ and $(\mathfrak{B}, \mathfrak{Z})$ are filtrators.

DEFINITION 1148. *Double filtrator of funcoids and reloids* is $(\mathsf{FCD}, \mathsf{RLD}, \mathbf{Rel})$.

DEFINITION 1149. $(\mathsf{FCD})f = \bigsqcap^{\mathfrak{A}} \operatorname{up}^{\mathfrak{Z}} f$ for $f \in \mathfrak{B}$.

DEFINITION 1150. $(\mathsf{RLD})_{\mathrm{out}} f = \bigsqcap^{\mathfrak{B}} \operatorname{up}^{\mathfrak{Z}} f$ for $f \in \mathfrak{A}$.

DEFINITION 1151. If (FCD) is a lower adjoint, define $(\mathsf{RLD})_{\mathrm{in}}$ as the upper adjoint of (FCD).

9.8.1. Embedding of $\mathfrak{A}$ into $\mathfrak{B}$. In this section we will suppose that (FCD) and $(\mathsf{RLD})_{\mathrm{in}}$ form a Galois surjection, that is $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f = f$ for every $f \in \mathfrak{A}$.

Then $(\mathsf{RLD})_{\mathrm{in}}$ is an order embedding from $\mathfrak{A}$ to $\mathfrak{B}$.

9.8.2. One more core part. I this section we will assume that (FCD) and $(\mathsf{RLD})_{\mathrm{in}}$ form a Galois surjection and equate $\mathfrak{A}$ with its image by $(\mathsf{RLD})_{\mathrm{in}}$ in $\mathfrak{B}$. We will also assume $(\mathfrak{A}, \mathfrak{Z})$ being a filtered filtrator.

PROPOSITION 1152. $(\mathsf{FCD})f = \operatorname{Cor}^{\mathfrak{A}} f$ for every $f \in \mathfrak{B}$.

PROOF. $\operatorname{Cor}^{\mathfrak{A}} f = \bigsqcap^{\mathfrak{A}} \operatorname{up}^{\mathfrak{A}} f \sqsubseteq \bigsqcap^{\mathfrak{A}} \operatorname{up}^{\mathfrak{Z}} f = (\mathsf{FCD})f$. But for every $g \in \operatorname{up}^{\mathfrak{A}} f$ we have $g = \bigsqcap^{\mathfrak{A}} \operatorname{up}^{\mathfrak{Z}} g \sqsupseteq \bigsqcap^{\mathfrak{A}} \operatorname{up}^{\mathfrak{Z}} f$, thus $\bigsqcap^{\mathfrak{A}} \operatorname{up}^{\mathfrak{A}} f \sqsupseteq \bigsqcap^{\mathfrak{A}} \operatorname{up}^{\mathfrak{Z}} f$. □

EXAMPLE 1153. $(\mathsf{FCD})f \neq \operatorname{Cor}'^{\mathfrak{A}} f$ for the double filtrator of funcoids and reloids.

PROOF. Consider a nontrivial ultrafiler a and the reloid $f = \mathrm{id}^{\mathsf{RLD}}_a$. $\operatorname{Cor}'^{\mathfrak{A}} f = \operatorname{Cor}'^{\mathsf{FCD}} \mathrm{id}^{\mathsf{RLD}}_a = \bigsqcup^{\mathsf{FCD}} \operatorname{down}^{\mathsf{FCD}} \mathrm{id}^{\mathsf{RLD}}_a = \bigsqcup^{\mathsf{FCD}} \emptyset = \bot^{\mathsf{FCD}} \neq a \times^{\mathsf{FCD}} a = (\mathsf{FCD})\, \mathrm{id}^{\mathsf{RLD}}_a$. □

I leave to a reader's exercise to apply the above theory to complete funcoids and reloids.