Conjecture 1035. A reloid f is monovalued iff

$$\forall g \in \mathsf{RLD}(\operatorname{Src} f, \operatorname{Dst} f) : (g \sqsubseteq f \Rightarrow \exists \mathcal{A} \in \mathscr{F}(\operatorname{Src} f) : g = f|_{\mathcal{A}}).$$

8.7. Complete reloids and completion of reloids

Definition 1036. A *complete* reloid is a reloid representable as a join of reloidal products $\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} b$ where $\alpha \in A$ and b is an ultrafilter on B for some sets A and B.

Definition 1037. A *co-complete* reloid is a reloid representable as a join of reloidal products $a \times^{\mathsf{RLD}} \uparrow^A \{\beta\}$ where $\beta \in B$ and a is an ultrafilter on A for some sets A and B.

I will denote the sets of complete and co-complete reloids from a set A to a set B as $\mathsf{ComplRLD}(A, B)$ and $\mathsf{CoComplRLD}(A, B)$ correspondingly and set of all (co-)complete reloids (for small sets) as $\mathsf{ComplRLD}$ and $\mathsf{CoComplRLD}$.

Obvious 1038. Complete and co-complete are dual.

Theorem 1039. $G \mapsto \bigsqcup\left\{\frac{\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} G(\alpha)}{\alpha \in A}\right\}$ is an order isomorphism from the set of functions $G \in \mathscr{F}(B)^A$ to the set $\mathsf{ComplRLD}(A, B)$.

The inverse isomorphism is described by the formula $G(\alpha) = \operatorname{im}(f|_{\uparrow\{\alpha\}})$ where f is a complete reloid.

Proof. $\bigsqcup\left\{\frac{\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} G(\alpha)}{\alpha \in A}\right\}$ is complete because $G(\alpha) = \bigsqcup \operatorname{atoms} G(\alpha)$ and thus

$$\bigsqcup\left\{\frac{\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} G(\alpha)}{\alpha \in A}\right\} = \bigsqcup\left\{\frac{\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} b}{\alpha \in A, b \in \operatorname{atoms} G(\alpha)}\right\}$$

is complete. So $G \mapsto \bigsqcup\left\{\frac{\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} G(\alpha)}{\alpha \in A}\right\}$ is a function from $G \in \mathscr{F}(B)^A$ to $\mathsf{ComplRLD}(A, B)$.

Let f be complete. Then take

$$G(\alpha) = \bigsqcup\left\{\frac{b \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} f)}}{\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} b \sqsubseteq f}\right\}$$

and we have $f = \bigsqcup\left\{\frac{\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} G(\alpha)}{\alpha \in A}\right\}$ obviously. So $G \mapsto \bigsqcup\left\{\frac{\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} G(\alpha)}{\alpha \in A}\right\}$ is surjection onto $\mathsf{ComplRLD}(A, B)$.

Let now prove that it is an injection:

Let

$$f = \bigsqcup\left\{\frac{\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} F(\alpha)}{\alpha \in A}\right\} = \bigsqcup\left\{\frac{\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} G(\alpha)}{\alpha \in A}\right\}$$

for some $F, G \in \mathscr{F}(B)^A$. We need to prove $F = G$. Let $\beta \in \operatorname{Src} f$.

$$f \sqcap (\uparrow^A \{\beta\} \times^{\mathsf{RLD}} \top^{\mathscr{F}(B)}) = \text{(theorem 607)}$$
$$\bigsqcup\left\{\frac{(\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} F(\alpha)) \sqcap (\uparrow^A \{\beta\} \times^{\mathsf{RLD}} \top^{\mathscr{F}(B)})}{\alpha \in A}\right\} =$$
$$\uparrow^A \{\beta\} \times^{\mathsf{RLD}} F(\beta).$$

Similarly $f \sqcap (\uparrow^A \{\beta\} \times^{\mathsf{RLD}} \top^{\mathscr{F}(B)}) = \uparrow^A \{\beta\} \times^{\mathsf{RLD}} G(\beta)$. Thus $\uparrow^A \{\beta\} \times^{\mathsf{RLD}} F(\beta) = \uparrow^A \{\beta\} \times^{\mathsf{RLD}} G(\beta)$ and so $F(\beta) = G(\beta)$.

We have proved that it is a bijection. To show that it is monotone is trivial.